III. Radial Diffusion

III.1 Violation of the Third Invariant

Whereas pitch-angle diffusion is customarily invoked as a loss mechanism for the radiation belts, diffusion in $\Phi$ is usually associated with creation of the belts. This is especially true of radial diffusion in which $M$ and $J$ are conserved, since particles then gain energy in the process of diffusing toward the earth from an external source (see below). Diffusion in $\Phi$ (radial diffusion) at constant $M$ and $J$ thus plays the dual role of injecting particles into the magnetospheric interior and accelerating the particles thereby injected to the energies observed.

In addition to particles that have entered from interplanetary space (and perhaps from the geomagnetic tail), the magnetosphere also contains protons and electrons born internally through the decay of albedo neutrons ejected from the upper atmosphere by energetic ($\gtrsim 100 \text{ MeV}$) solar protons and galactic cosmic-ray particles colliding inelastically (in the nuclear sense) with gas atoms. These internal source mechanisms are known as SPAND and CRAND, respectively, for solar-proton (and cosmic-ray) albedo neutron decay. These sources (CRAND is about ten times as intense a particle source as SPAND) typically account for the presence of energetic protons and electrons in the inner zone, but radial diffusion plays an essential role in bringing about the observed spatial and spectral distribution of these particles [38]. In addition, radiation-belt particles may possibly experience in situ acceleration to high energies [44] through the absorption of plasma-wave energy. Such an event might easily be interpreted as an “injection” of the energetic particles into the magnetospheric interior (see Section IV.6).

Artificial radiation belts created by high-altitude nuclear detonations (1958—1963) once contributed substantially to the inner-zone particle population. These artificial belts, which had decayed to an intensity below that of the natural radiation by the year 1968, yielded some of the earliest measurements of a radial-diffusion coefficient for radiation-belt electrons in the magnetosphere.

In the outer zone, radial diffusion plays an all-important role in maintaining the level of trapped radiation. Direct observational evidence for the occurrence of third-invariant violation in the outer zone is shown in Fig. 23, which is a tracing of data obtained by instruments
Fig. 23. Drift-periodic echoes in outer-zone electron fluxes, as observed on ATS I following a negative magnetic impulse [35] at 2330 UT (1330 LT).

on the geosynchronous equatorial satellite ATS I (longitude 150° W), together with the magnetogram (horizontal, or H, component) for the same time period (1300—1400 local time) from the ground-based station at Honolulu. The interpretation of Fig. 23 is that a negative magnetic impulse, presumably caused by a sudden decrease in solar-wind pressure at the magnetopause, propagates inward from the magnetopause and arrives at Honolulu several minutes after encountering the spacecraft.\(^\text{20}\)

Upon arrival at synchronous altitude, the impulse causes a simultaneous decrease of the electron flux observed in each of the seven energy channels. As time goes on, however, particles near the satellite at the arrival time of the impulse drift toward the night side, and electrons from the night side (where the negative impulse was less severe) drift to the azimuthal position of the satellite. This accounts for the recovery of the fluxes in each channel on a time scale of half the energy-dependent drift period. The relative minimum in flux recurs with the return (to the day side) of those particles most severely influenced by the impulse.

\(^{20}\)This delay time is in accord with the time required for a rarefractional (magnetosonic) impulse to travel the required distance of 5.6 earth radii at approximately the Alfvén speed.

These drift-periodic echoes in the outer-zone electron flux persist well after the passage of the magnetic impulse initiating them. Moreover, the fact that each energy channel “oscillates” at its own characteristic drift frequency is convincing evidence for drift-phase organization of the particles, which therefore (cf. Section II.1) have been dispersed with respect to \(\phi\) \((\approx 2\pi a^2 B_0/L)^{\text{21}}\). The nonvanishing energy bandwidth of each detection channel corresponds to a drift-frequency bandwidth that thoroughly phase-mixes the observations on a time scale of three or four drift periods. Particles initially differing in both \(\phi\) and energy retain their separate identities, but the detectors can no longer distinguish among them.

The practical fact of phase mixing, and the fact that consecutive sudden impulses are statistically uncorrelated on the drift time scale, provide the essential degree of randomness that makes it appropriate to speak of third-invariant violation in terms of diffusion with respect to \(\phi\). At constant \(M\) and \(J\), i.e., with pitch-angle diffusion neglected, the radial-diffusion equation

\[
\frac{\partial T}{\partial t} = \frac{\partial}{\partial \phi} \left[ D_{\phi\phi} \frac{\partial T}{\partial \phi} \right] = L^2 \frac{\partial}{\partial \phi} \left[ \frac{1}{L^2} \frac{\partial T}{\partial \phi} \right]
\]

follows directly from (2.01), since \(D_{\phi\phi}(dL/d\phi)^2\). The distribution function \(f\) is equal to \(J_L/p^2\), evaluated on a surface generated by the mirror points of ions or electrons having in common their values of \(M\) and \(J\).

In a dipole field this surface coincides with the equatorial plane \((\theta=\pi/2)\) for particles having \(J=0\). For \(J\neq0\) the mirror-point surface satisfies the equation

\[
|y|^{2} = 8 m_0 B_0 a^2 (M/J^2 L) = B_0 a^2 K^2 L,
\]

where \(y\) is related by (1.25) to the mirror colatitude \(\theta_m\). With the aid of (3.02) and (1.31), the variation of \(y\) with \(L\) at constant \(M\) and \(J\) is plotted in Fig. 24 for selected values of \(y\) (the value of \(y\) at \(L=7\)). The \(L=7\) shell is often used as a reference location in radiation-belt theory because it is quite near the outer boundary of stable trapping, and therefore adjacent to a possibly important source of moderately energetic particles (i.e., solar cosmic rays that have entered the magnetosphere). A secondary reason for the popularity of \(L=7\) as a reference shell [40] is that an equatorially mirroring particle's nonrelativistic...
III. Radial Diffusion

Fig. 24. Systematic variation of $y$ (sine of equatorial pitch angle) with $L$ at constant $M$ and $J$, applicable to radial diffusion caused by magnetospheric impulses.

The first two invariants while violating $\phi$. Such mechanisms may involve particle collisions or bounce- and cyclotron-resonant interactions with magnetospheric waves. Radial diffusion mechanisms that violate $M$ and/or $J$ often lack the ability to energize particles efficiently in the process, and they generally play a less certain role than sudden impulses in the overall picture of radiation-belt dynamics.

### III.2 Magnetic Impulses

In the magnetic-field model specified by (1.45), sudden impulses in $B$ correspond to sudden changes in $b$, the geocentric stand-off distance to the subsolar point on the magnetopause. The stand-off distance $b$ is governed, according to (1.43), by the momentum flux of the solar wind. An encounter with the plasma ejected by a solar flare, for example, can lead to a sudden contraction and/or expansion of the magnetosphere. A decrease in $b$ that is sudden on the drift time scale represents a sudden contraction of the magnetosphere. This contraction consists of both an azimuthally symmetric compression of $B$ (the $B_1$ term) and an azimuthally asymmetric distortion of $B$ (the $B_2$ term). The symmetrical compression, which is easily identified from the magnetograms of ground-based ($r = a$) observatories, is adiabatic to the trapped particles. All drift phases $\phi_3$ respond identically to the symmetrical part of the sudden impulse, and so this part is reversible. It conserves $\phi$ and produces no radial diffusion.

**Induced Electric Field.** The accompanying asymmetric distortion (the $B_2$ term) is not easily distinguished at $r = a$, where it is small in magnitude. However, this part of the impulse does violate the third invariant, thereby producing drift echoes (Fig. 23) and radial diffusion. A sudden impulse in $B$ affects the geomagnetically trapped particles by virtue of an induced electric field $E$, which may be calculated term by term from a field expansion [29] of the form (cf. (1.46))

$$ E_r(r, \theta, \varphi, t) = \sum_{lmn} E_{lm}(l, m, n; t)(r/b)^l \sin^l \theta \sin m \varphi $$  \hspace{1cm} (3.03a)

$$ E_\theta(r, \theta, \varphi, t) = \sum_{lmn} E_{\theta lm}(l, m, n; t)(r/b)^l \cos \theta \sin^l \theta \sin m \varphi $$  \hspace{1cm} (3.03b)

$$ E_\varphi(r, \theta, \varphi, t) = \sum_{lmn} E_{\varphi lm}(l, m, n; t)(r/b)^l \sin^l \theta \cos m \varphi. $$  \hspace{1cm} (3.03c)

If the Maxwell relation $\mathbf{E} \times \mathbf{B} = - \partial \mathbf{B} / \partial t$, written out in its three components, is applied to (1.46) and (3.03), the time-dependent (but position-independent) coefficients of $[(r/b)^l \sin^l \theta \sin m \varphi]$ and $[(r/b)^l \cos \theta$
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\( \times \sin^2 \theta \sin m \varphi \) can be isolated to yield the relationships

\[
(n + 1) E_\varphi(l, m, n; t) = m E_\varphi(l + 1, m, n; t)
+ \frac{(\alpha/c)(b/a)^n}{(b/a)^n} B_\varphi(l, m, n; t) \tag{3.04a}
\]

\[
(n + 1) E_\varphi(l, m, n; t) = \frac{(l + 1)}{l} E_\varphi(l + 1, m, n; t)
- \frac{(\alpha/c)(b/a)^n}{(b/a)^n} B_\varphi(l, m, n; t). \tag{3.04b}
\]

where \( B = \frac{\partial B}{\partial t} \). The third component of \( \mathbf{cV} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \) is redundant, since \( \mathbf{V} \cdot \mathbf{B} = 0 \).

One more condition on \( \mathbf{E} \) must be specified in order to solve (3.04). It is customary to state this subsidiary condition as \( \mathbf{E} \cdot \mathbf{B} = 0 \) [32]. Such a statement is usually justified by an appeal to the cold plasma which is assumed to fill the magnetosphere. The cold plasma serves to short-circuit each field line, in which case the impulsively expanding \( (db/\partial t > 0) \) or contracting \( (db/\partial t < 0) \) magnetospheric medium is governed by the laws of magnetohydrodynamics (MHD). Since the impulse therefore propagates through the magnetosphere at approximately the Alfvén speed, the field model summarized by (1.46) admittedly violates the principle of causality on time scales shorter than \( \sim c/e_a \). For drift periods exceeding a few minutes, however, the arrival time of the impulse at any \( L \) shell is practically independent of \( \varphi_\theta \), and this condition permits the simplified (instantaneous-response) model to be used for the time-varying \( \mathbf{B} \) field.

For the magnetic-field model given by (1.48), application of \( \mathbf{E} \cdot \mathbf{B} = 0 \) to (3.04) yields the recursion relation [29]

\[
[(2n + l + 2)/(n + 1)] E_\varphi(l, m, n; t)
= (B_\varphi/B_\theta)^{(l + l - 2)/(n - 2)} E_\varphi(l, m, n - 3; t)
+ (B_\varphi/B_\theta)^{(l - n + 2)/(n - 3)} E_\varphi(-l - 1, m, 1, n - 4; t)
+ (B_\varphi/B_\theta)^{(l - n - 2)/(n - 3)} E_\varphi(-l + 1, m, 1, n + 4; t)
- (B_\varphi/B_\theta)^{(l + m + 2)/(n - 3)} E_\varphi(l + 1, m, 1, n - 4; t)
- (B_\varphi/B_\theta)^{(l + m - 2)/(n - 3)} E_\varphi(l - 1, m, 1, n + 4; t)
- (4/3) B_\varphi (B_\varphi/B_\theta)^{(l + m + 2)/(n - 3)} E_\varphi(l + 1, m, 1, n + 4; t)
- (4/3) B_\varphi (B_\varphi/B_\theta)^{(l + m - 2)/(n - 3)} E_\varphi(l - 1, m, 1, n - 4; t)
- (1/6) B_\varphi (B_\varphi/B_\theta)^{(l + m + 2)/(n - 3)} E_\varphi(l + 1, m, 1, n - 4; t)
- (1/6) B_\varphi (B_\varphi/B_\theta)^{(l + m - 2)/(n - 3)} E_\varphi(l - 1, m, 1, n + 4; t) \tag{3.05}
\]

where the Kronecker symbol \( \delta_{ij} \) is equal to unity (rather than zero) only if \( i = j \). Closure of the recursion formula is achieved by requiring that the \( \mathbf{E} \) induced by \( db/\partial t \) remain finite in the limit \( t = 0 \). This requirement forces \( E_\varphi(l, m, n; t) \) to vanish if \( n < 0 \). From this starting point it is possible to generate all the coefficients \( E_\varphi(l, m, n; t) \) by means of (3.05). The nonvanishing \( E_\varphi(l, m, n; t) \) of lowest order \( n \) is \( E_\varphi(1, 1, 2; t) = -4/7c)(a/b)^3(db/\partial t)B_\varphi \).

### III.2 Magnetic Impulses

The recursion relation yields definite algebraic values for several coefficients which have \( m = 0 \), and which therefore ostensibly can have no physical significance [see (3.03a)]. Such coefficients that multiply zero (in the form of \( \sin m \varphi \)) are ignored. A correct implementation of (3.05) thus leads to a unique set of electric-field coefficients \( E_\varphi(l, m, n; t) \), where \( m = 0 \), from which \( E_\varphi(l, m, n; t) \) and \( E_\varphi(l, m, n; t) \) are obtainable by means of (3.04). All nonvanishing electric-field coefficients for which \( n \leq 10 \) are listed in Table 7.

<table>
<thead>
<tr>
<th>Table 7. Electric-Field Coefficients</th>
</tr>
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<tbody>
<tr>
<td>( E_\varphi(1, 1, 2; t) = (4/7)B_\varphi(a/b)^3 )</td>
</tr>
<tr>
<td>( E_\varphi(1, 1, 3; t) = -(9/10)B_\varphi(a/b)^3 )</td>
</tr>
<tr>
<td>( E_\varphi(2, 2, 6; t) = (1/6)B_\varphi(a/b)^3 )</td>
</tr>
<tr>
<td>( E_\varphi(1, 1, 4; t) = -(1/70)B_\varphi(a/b)^3 )</td>
</tr>
<tr>
<td>( E_\varphi(2, 2, 8; t) = (25/273)B_\varphi(a/b)^3 )</td>
</tr>
<tr>
<td>( E_\varphi(1, 1, 10; t) = -(1/483)B_\varphi(a/b)^3 )</td>
</tr>
<tr>
<td>( E_\varphi(2, 2, 10; t) = -(1/70)B_\varphi(a/b)^3 )</td>
</tr>
</tbody>
</table>

Response of Trapped Particles. This analytical representation of the \( \mathbf{E} \) field induced by a time-varying model \( \mathbf{B} \) field is especially useful for following the response of trapped particles to a magnetic impulse. Each particle experiences an electric drift at velocity

\[
\mathbf{v}_d = (c/B^2) \mathbf{E} \times \mathbf{B} \tag{3.06}
\]

in addition to its gradient, curvature, and other (cf. Section III.6) drifts. As a consequence, the particle may change its value of \( L = 2 \pi a^2 B_\varphi \).
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Except for particles mirroring at the equator, the bounce average required in applying (3.06) is quite onerous. A rather different approach, based on (1.77b), is more expedient for calculating the radial-diffusion coefficient $D_{rL}$ to lowest order in $(B_2/B_0)$, for arbitrary mirror latitude.

The more expedient approach is based on the fact that $v_\phi$ as given by (3.06), can be identified as the local velocity of a field line if the E field induced by $\partial \mathbf{B}/\partial t$ is everywhere perpendicular to $\mathbf{B}$. In other words, if only the $\mathbf{E} \times \mathbf{B}$ drift is considered, the particle remains on its original field line, as identified by the label $L_D$. The proof that field-line motion can be traced in this manner follows from the identity

$$B^2(d L_d/dt) = B^2 \mathbf{L}_d \times \mathbf{E} + B^2 v_\phi \mathbf{V} L_d$$

$$= B^2 \mathbf{L}_d \times \mathbf{E} + B^2 v_\phi \mathbf{V} L_d = 0,$$  

(3.07)

which can be verified with the aid of Table 7 to each order in $e_1$ and $e_2$ (see Section 1.7). The degree of accuracy inherent in (1.69b), which implies

$$L_d \approx (r/a \sin^2 \theta)[1 + (B_2/2 B_0)(r/b)^3 - (2 B_2/21 B_0 \sin \theta)(r/b)^4(7 \sin^2 \theta - 3) \cos \varphi],$$

(3.08)

is adequate to verify (3.07) in first and second order. A more extensive proof (to higher order) is not required here, but could easily be generated [32].

Whereas the $\mathbf{E} \times \mathbf{B}$ drift induced by a time variation of $b$ yields no immediate change in a particle's $L_d$ coordinate, the gradient-curvature drift does. According to (1.77b) this change is of the form

$$d L_d/dt = -\varphi (B_2/252 B_0) L_d^2 (a/b)^4 [Q(y)/D(y)] \sin \phi.$$  

(3.09)

The coordinate $L_d$ to which the particle would return at $\varphi = \pm \pi/2$ in a magnetosphere frozen in time ($db/dt = 0$) properly labels the drift shell in the sense that.

III.2 Magnetic Impulses

$$|\Phi| = \int_0^\infty \int_0^{2\pi} B_0(a/r)^3 r^2 dr d\varphi$$

$$- \int_0^{2\pi} \int_0^\infty \left[ B_1(a/r)^3 - B_2(a/r)^4(r/b) \cos \varphi \right] r^2 dr d\varphi$$

$$= \frac{2 \pi a^2 B_0 L_d(e_2)}{1 + O(e_2^2) + \cdots}.$$  

(3.10)

For convenience, the third invariant has been evaluated using the equatorial plane $(e = \pi/2)$ of the magnetosphere, in which case $B$ points in the $-\hat{r} \hat{\theta}$ direction and has a magnitude given by (1.45b). Since the end result of (3.10) has no correction terms of order $e_1$ or $e_2$, the definition $L = L_d(\pm \pi/2)$ suffices for a calculation of $D_{rL}$ to lowest order.

**Diffusion Coefficient.** Since $L = L_d(\pm \pi/2)$, it follows from (1.77b) that

$$L = L_d [1 - (B_2/252 B_0) L_d(a/b)^4 [Q(y)/D(y)] \cos \varphi].$$

(3.11)

The instantaneous shell parameter $L$ thus changes at a rate

$$d L_d/dt = (B_2/63 a B_0) L_d (a/b)^2 (d b/dt) [Q(y)/D(y)] \cos \varphi$$

(3.12)

to lowest order in $e_1$ and $e_2$. The radial diffusion coefficient $D_{rL} = (1/2 \pi)(\partial L/\partial t)$ is obtained by integrating (3.12) over an interaction time $t = r \Omega / Q_3$, during which $\cos \varphi = \cos Q_3 = \cos (Q_3 + \varphi_3)$. It is convenient to express the result in terms of $\delta_4(a/2\pi)$, which is defined as the spectral density function of $B_1(a/r)^3$. The procedure for obtaining $D_{rL}$ is much the same as that used in Section II.4, and the result [56, 57] is

$$D_{rL} = 2 Q_3^2 (B_2/756 B_0 Q_3) J_0(a/b)^2 [Q(y)/D(y)]^2 \delta_4(a/2\pi).$$

(3.13)

When radial diffusion is caused by magnetic impulses, the energy dependence of $D_{rL}$ is contained entirely in $Q_3$. If the impulses rise sharply and decay slowly (like a step function) on the drift time scale, then $\delta_4(a/2\pi)$ is proportional to $Q_3^2$ and all energy dependence disappears. At sufficiently high energies, the particle drift period becomes somewhat comparable to the rise time of an impulse (see Fig. 22), and this range of drift frequencies finds $\delta_4(a/2\pi)$ falling more sharply than $Q_3^2$. It follows that $D_{rL}$ ultimately decreases somewhat with increasing energy. At constant $M$ and $J$, however, the drift frequency decreases with increasing $L$. Thus, any inverse dependence of $D_{rL}$ on particle energy tends to strengthen the $L$ dependence of the radial-diffusion coefficient.

The dependence of $D_{rL}$ on equatorial pitch angle is contained primarily in the factor $[Q(y)/D(y)]^2$. This factor, as approximated within $\sim 1\%$ by means of (1.36) and (1.79), varies by nearly an order of magnitude.
between \( y=1 \) and \( y=0 \) [56, 57]. The function \( Q(y)/180 D(y) \) is plotted in Fig. 25. At energies sufficiently high that the drift period is comparable to the rise time of a magnetic impulse, this extreme variation of \( D_{LL} \) with \( x \) is slightly moderated by the fact that particles having \( x \approx 1 \) drift more slowly in azimuth than those for which \( x=0 \) [see (1.35)]. At a given energy, of course, this variation of \( \Omega_3 \) with \( x \) is very weak.

![Graph](image)

Fig. 25. Variation of \((D_{LL})^{1/2}\) with \( y \) for radial diffusion caused by magnetic sudden impulses. Data points have been determined by numerical computation [19]. Solid curve is analytical approximation [66] based on (1.36) and (1.79).

In summary, the radial-diffusion coefficient caused by magnetic impulses that rise sharply and decay slowly on the drift time scale is virtually independent of energy. For particles mirroring at the equator, the coefficient is given [29] by

\[
D_{LL} = 2 \Omega_3 (5 B_s/21 B_1 B_0)^2 L^{15} (a/h)^2 \mathcal{H}_2(\Omega_3/2\pi) \quad (3.14)
\]

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Since \( Q(1)=180 D(1) \), in the case that \( \mathcal{H}_2(\Omega_3/2\pi) \) falls off as \( \Omega_3^{-2} \), there is no energy dependence in \( D_{LL} \). Thus, the diffusion coefficient depends on \( y \) through the factor \([Q(y)/D(y)]^2 \) in (3.13), and on \( L \) through the factor \( L^{15} \). But since \( y \) and \( L \) are related via (3.02), the factor \([Q(y)/D(y)]^2 \) exhibits an inverse variation with \( L \). Except at \( y=0 \) and \( y=1 \), this factor tends to moderate the variation of \( D_{LL} \) with \( L \). With the aid of Fig. 25, it is possible to evaluate the ratio of \( D_{LL} \) at any \( L \) to \( D_{LL} \) at \( L=7 \) and \( y=1 \) for selected values of \( y \), under the assumption that \( \mathcal{H}_2(\omega/2\pi) \) is a constant. The results, plotted in Fig. 26, are principally of interest for high-energy protons and helium ions, which may in fact escape pitch-angle diffusion during their period of residence in the magnetosphere.

![Graph](image)

Fig. 26. Radial variation of \( D_{LL} \) driven by magnetic impulses, for selected values of \( y \) at \( L=7 \) (\( y=0, 0.4, 0.6, 1.0 \)). Dashed line \((y=0)\) is not realized in practice, because of the loss cone.

The magnitude of the spectral density function \( \mathcal{H}_2(\omega/2\pi) \) is likely to vary with magnetic activity (as measured by an index such as \( K_p \) or \( D_s \)), and so the otherwise arbitrary interaction time \( \tau \) used in this chapter is limited by the time scale of several days characteristic of changes in magnetospheric "weather". However, if the observations of \( f \) are sufficiently coarse-grained as to average over genuine temporal variations of \( D_{LL} \), it may be possible to identify a mean diffusion coefficient applicable to a much longer time interval (see Chapter V). Particle
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lifetimes in the inner proton belt are even long enough (see Fig. 14, Section II.2) to permit averaging $D_{LL}$ and $f$ over a large number of solar cycles.

III.3 Electrostatic Impulses

Abrupt temporal changes in the electrostatic potential associated with plasma convection are characteristic of geomagnetic storms and magnetospheric substorms. Impulses of this type can be represented by a time-dependent coefficient $E_c$ in (1.52). For particles of radiation-belt energy $W > l/g l$, the drift-shell asymmetry caused by the mean value of $E_c$ can be neglected in the calculation of $D_{LL}$ to lowest order in $q/E_c a L_d / W$. Moreover, the magnetic field is assumed to be given by (1.16). In this situation it is evident from (1.66) that $|\phi| = 2\pi a^2 B_0 / L_a(0)$, with no first-order correction in $q/E_c a L_d / W$. Thus, the instantaneous shell parameter $L$ is given in lowest order by

$$L = L_d [1 - (T(y)/D(y)) \sin \varphi]$$

in view of (1.35). Since (1.65a), to the order of accuracy inherent in (1.15), implies that

$$d L_d / d t = \phi (E_c / B_0)(c/\Omega_3 a) \sin \varphi$$

for a particle drifting in azimuth under the influence of (1.52), it follows that

$$d L_d / d t = -(E_c / B_0)(c/\Omega_3 a) \sin \varphi$$

for this particle. With $E_c$ represented as the sum of purely temporal Fourier components [cf. (2.28)], the particle selects that component for which $\omega = \Omega_3$ after an interaction time $\tau > 2\pi \Omega_3$.

The diffusion coefficient $D_{LL} \equiv (1/2\pi) \langle \Delta L^2 \rangle$ obtained from (3.17) by the methods of Section II.4 is given by

$$D_{LL} = 2(c/4a B_0)^2 L^2 \delta_c(\Omega_3/2\pi),$$

where $\delta_c(\omega/2\pi)$ is the spectral density function of $E_c$ (see Section I.6). A particle's energy and equatorial pitch angle enter (3.18) only via $\Omega_3$. For electrostatic impulses that rise sharply and decay slowly on the drift time scale, the spectral density $\delta_c(\Omega_3/2\pi)$ falls as $\Omega_3^{-2}$. In this case, the functional form of $D_{LL}$ is

$$D_{LL} = 2(q/24 B_0)^2 [T(y)/D(y)](c^2/M)^2$$

$$\times [1 + (2M B_0/m_0 c^2 \gamma L_L)^2] \omega^2 \delta_c(\omega/2\pi),$$

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where $\omega/2\pi$ is any frequency whose reciprocal lies well between the rise and decay times of the typical electrostatic impulse.

It is conventional to compare spatially coincident magnetospheric fluxes of different ionic species at common kinetic energy per nucleon, i.e., at common particle velocity. This convention greatly simplifies the comparative analysis of collisional effects (see Section II.2). The ions of interest are typically nonrelativistic, and so the comparison applies essentially at common $y$, $L$, and $M/A$, where $A$ is the number of nucleons in the ion. The respective electrostatic radial-diffusion coefficients thus scale as $(q/A)^2$. The magnitudes of $D_{LL}$ for $\text{H}^+ : \text{He}^++ : \text{He}^+$ therefore scale as 16:4:1. When coupled with the expectation that magnetospheric helium nuclei (originally interplanetary alpha particles) spend up to half their radiation-belt lifetimes as $\text{He}^+$ by virtue of charge exchange (see Section II.2), this property of electrostatic diffusion provides a possibly interesting explanation [40] for the observational fact (see Section IV.5) that ratios of helium-ion flux to proton flux (often abbreviated $\alpha/p$ and $\text{He}^+/p$) at common $E/A$ in the magnetosphere (well off the equator) are orders of magnitude smaller than the $\alpha/p$ ratio in the solar wind [59].

Among particles of the same species, the diffusion coefficient given by (3.19) is rather sensitive to particle energy, but notably insensitive

Fig. 27. Radial variation of $D_{LL}$ driven by electrostatic impulses, for nonrelativistic particles having a common value of $E$ at $L = 7$ and selected values of $y$ at $L = 7$ ($\gamma = 0, 0.2, 0.4, 0.6, 0.8, 1.0$). Dashed line ($\gamma = 0$) is not realized in practice.
III. Radial Diffusion

to equatorial pitch angle. Neither energy nor pitch angle is invariant during radial diffusion at constant \( M \) and \( J \), however. Thus, a proper comparison should follow the spirit of Fig. 26, wherein particles are distinguished according to their values of \( y \) at \( L = 7 \) in a dipole field.

In electrostatic diffusion it is logical to compare particles having a common kinetic energy \( E_7 \) at \( L = 7 \), i.e., a common value of \( M/y^3 \).

The derivation of (3.19) assumes that \( \alpha^2 \beta_\alpha(\alpha/2\pi) \) is constant for all frequencies of interest. If attention is limited to nonrelativistic particles, such as radiation-belt ions, the variation of \( D_{LL} \) with \( L \) is that of \( L^{10}[T(v)/D(v)]^2(y\gamma)^2 \). With the aid of Fig. 24, which indicates the variation of \( y \) with \( L \) for selected values of \( y \), the ratio of \( D_{LL} \) at arbitrary \( L \) to \( D_{LL} \) at \( L = 7 \) and \( y = 1 \) has been evaluated for these selected values of \( y \). The result is shown in Fig. 27. The common value of \( E_7 \) that forms the basis of this comparison is assumed to be such that 270 \( E \sim \ell_{\gamma 0} c^2 \) in order to justify using the nonrelativistic form of (3.19) down to \( L \approx 1.08 \), where the dense atmosphere terminates the inner belt (see Section II.2).

Spectral Density. Extrapolation of (3.19) to ring-current energies and below is forbidden on a variety of grounds. Consider (for example) a random sequence of impulses, each consisting of an ideally instantaneous jump from \( E_0 \) to \( E_0 + \Delta E_0 \), followed by an exponential decay to \( E_0 \) with an \( e \)-folding time \( \tau_d \). The spectral density \( \beta_\alpha(\alpha/2\pi) \) is then given \(^{23} \)

\[
\beta_\alpha(\alpha/2\pi) = \frac{2(2\pi/\alpha)^2 \Sigma(\Delta E_0)^2}{1 + \alpha^2 \tau_d^2},
\]

where \( \alpha > 0 \) and \( \Sigma(\Delta E_0)^2 \) denotes the sum of the squares of all sudden increments in \( E_0 \) initiated within an arbitrarily long (but statistically homogeneous) time interval of duration \( \tau \). The validity of (3.19) thus requires \( \alpha^2 \tau_d \gg 1 \) at \( \alpha = |\Omega_3| \). It is presumed that \( \tau_d \approx 2 \text{hr} \), and that most radiation-belt particles therefore comply with the conditions of (3.19). At ring-current (hot-plasma) energies and below, it is essential to reconsider the radial-diffusion problem in terms of (1.65), without making the simplifying approximation that \( W \gg |q| V_0 \).

On the other hand, the spectral density \( \beta_\alpha(\Omega_3/2\pi) \) falls more rapidly than \( \Omega_3^{-2} \) for particles having drift periods comparable to or smaller than the rise time of a typical impulse. The approximation of a vanishing rise time, as used in (3.20), is appropriate only if the particles of interest

\(^{23}\)The spectral density function given by (3.20) has been constructed in a manner easily generalized to other types of impulses. For example, if \( E_0 \) is replaced by \( B_i |B| \) in (3.20), the result is a formula for \( \beta_\alpha(\alpha/2\pi) \), the spectral density function for magnetic sudden impulses (Section III.2).

have drift periods well in excess of the true rise time. Moreover, the magnitude of \( \beta_\alpha(\alpha/2\pi) \) is likely to vary with the level of geomagnetic activity, as measured by an index such as \( K_p \) or \( D_a \) (cf. Section III.2).

III.4 Bounce Resonance

Resonance of an MHD or electrostatic wave with harmonics of a particle's bounce frequency has been invoked previously (see Section II.4) as a mechanism for pitch-angle diffusion. There it was noted that confinement of the electric-field perturbation \( e \) to a meridional plane would prevent contamination by radial-diffusion effects. Conversely, a component of \( e \) in the azimuthal direction provides for the possibility of radial diffusion. The case of an electrostatic wave (for which \( e \) is parallel to \( k \)) propagating purely in the azimuthal direction (see also Section III.5) is essentially covered by (3.21) and (3.22). The resonance condition is found to be \( \omega = m \Omega_3 \). Even if \( m \) is a very large number \( (-e^{-1}) \) for a resonant particle), the resonance condition is unaffected by the bounce motion if \( k \) is everywhere normal to \( B \) for an electrostatic wave.
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If, however, the electrostatic wave is such that field lines are not equipotentials, e.g., as in (2.37), then the condition for resonance takes on the form \[ \omega - m \Omega_1 - l \Omega_2 = 0. \] (3.23)

Just as in (2.33), the various values of \( l \) enter the diffusion coefficient weighted by \( J_f (k \Omega / 2 \Omega_1, \Omega_2) \), which is small if the order \( (l) \) is much larger than the argument \( (k \Omega / 2 \Omega_1, \Omega_2) \). If \( m \ll |\omega|^{-1} \), it may be instructive to define an azimuthal wavenumber \( k_\phi = (m \Omega / \Omega_2) \), and a bounce-averaged particle drift velocity \( v_\phi = \Omega_2 \sin \theta \) in the dipole field. The resonance condition then reads \[ \omega - k_\phi v_\phi = l \Omega_2, \] (3.24)

which may be interpreted as a Doppler-shifted bounce resonance by analogy with (3.28). On the other hand, if \( |m \Omega| \gg |\Omega_2| \), it may be instructive to view (3.23) as a bounce-modified drift resonance. Since which may be interpreted as a Doppler-shifted bounce resonance by analogy with (3.28). On the other hand, if \( |m \Omega| \gg |\Omega_2| \), it may be instructive to view (3.23) as a bounce-modified drift resonance. Since the two interpretations are fully equivalent for any \( m \) and \( l \), however, the connection with radial diffusion (Section III.3) is quite evident.

A similar connection may be drawn between the magnetic impulses of Section III.2 and an MHD wave propagating partly in the direction of \( \mathbf{V} \mathbf{L} \), and partly in the directions of \( \mathbf{B} \) and \( \mathbf{p} \). The electric-field perturbation \( e \) for an MHD mode is normal to \( \mathbf{k} \) and \( \mathbf{B} \) (in the cold-plasma approximation), and thus lies in the plane of \( \mathbf{V} \mathbf{L} \), and \( \mathbf{p} \). The \( \mathbf{p} \) component of \( e \) leads to radial diffusion, the \( \mathbf{B} \) component of \( h \) to resonance, and the \( \mathbf{B} \) component of \( k \) to bounce resonance. The condition imposed by (3.23) includes both bounce resonance and drift resonance. Either can be isolated by assigning \( m = 0 \) or \( l = 0 \), respectively.

III.5 Cyclotron Resonance

Because it leads to substantial pitch-angle diffusion, the Doppler-shifted \((k_1, v_0)\) cyclotron resonance considered in Section II.5 is principally a loss mechanism for geoelectronically trapped particles. Cyclotron resonance is not known to be an important mechanism for radial diffusion in the radiation belts. A particle is perhaps displaced by one gyroradius in the course of diffusing by one radian in pitch angle. In the absence of shell splitting (see Sections I.7 and III.7), the resulting radial-diffusion coefficient is of order \( e^2 L^2 D_{sc} \). This is rather insignificant for radiation-belt \((|\omega| \approx 1 \); Section I.1) particles, since the root-mean-square displacement in \( L \) is only of order \( e L \) during the lifetime of a particle (in weak diffusion; see Section I.7). The most energetic radiation-belt ions (for which \( |\omega| \) is nearest to unity) tend to deposit their energy in the tenuous atmosphere without significant change of pitch angle.

The conditions under which radial diffusion might occur by virtue of cyclotron resonance are quite different from the conditions explored in Section II.5. Consider, for example, a wavelike electrostatic potential of the form

\[ V_0(r, \theta, \phi, t) = a E_\phi (L_{\phi}) \sin(m \theta - \omega t + \phi_m), \] (3.25)

where \( \omega / 2 \pi \) is of the order of a particle's gyrofrequency. A wave of this type may be generated by virtue of an unstable spatial gradient of \( \phi \), e.g., \( \partial \phi / \partial L < 0 \), with the conserved quantities held constant. Such an azimuthally propagating wave is called a drift wave, whether or not cyclotron resonance is involved.

The unperturbed motion of an equatorially mirroring particle may be represented by

\[ \phi = \Omega_3 t + \phi_3 + \frac{m c q B L_{\phi}}{2 \omega} \sin(\Omega_1 t + \phi_1). \] (3.26)

The postulated drift wave does not alter the equatorial pitch angle \((\pi/2)\) of such a particle. The particle's interaction with the wave specified by (3.25) yields a \( \phi_3 \)-dependent drift in \( L \) (= \( L_{\phi} \) in a dipole field) given by

\[ dL/dt = -c/a |L_{\phi}| \sum J_{l}(-m v_\phi/\Omega_1 L_{\phi}) \cos[(\omega - \Omega_1 - m \Omega_3) t - (\phi_m + \phi_1 + m \phi_2)]. \] (3.27)

The resonance condition \( \omega = \omega_m = \Omega_1 + m \Omega_3 \) leads to a radial-diffusion coefficient of the form

\[ D_{sc} = 2 c a B L_{\phi}^2 \sum J_{l}^2(m v_\phi/\Omega_1 L_{\phi}) \delta_{lm}(L_{\phi}/2 \pi), \] (3.28)

where \( \delta_{lm}(L, \omega/m) \) is the spectral density of all waves having the form of (3.25). Note that \( m \) represents an azimuthal index, not a mass, in (3.25)–(3.28). The leading factors in (3.22) and (3.28) differ only because each term in (3.21) is a superposition of two waves having the form of (3.25).

In the argument of the Bessel function \( J_{l} \), the factor \( m / L_{\phi} \) plays the role of \( k_\phi \) in (3.24) or of \( k_L \) in (2.43). Thus, for azimuthal wavelengths comparable to a particle's gyroradius, a drift wave can resonate with the gyration of the particle in a manner that leads to radial diffusion. The pitch-angle of an equatorially mirroring particle is unaffected by this relation, but the energy of a resonant particle changes in accordance with the relation

\[ d(p^2)/dL = 2 q p - e(dL/dt)^{-1} = 2 q m^2 B L_{\phi} m_{\phi} \omega/m c L_{\phi} \] (3.29)

in a localized region where \( dE_{\phi}/dL = 0 \). The ratio \( m/q \) is negative for a wave \((m)\) propagating in the direction of the resonant particle's azi-
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muthal drift. Thus, an outward diffusive flow of trapped particles (arising from an inward gradient of \( f \) with respect to \( L \)) leads to a transfer of particle energy to the wave\(^{24}\). The interaction evidently conserves

\[
p^2 - 2(q/m)(r^2 B_0 m_0/c)\left(\frac{\eta_0 L^2}{\sqrt{c}}\right) dL = \text{constant},
\]

and so this is the quantity that must be held constant in evaluating \( \partial f/\partial L \). Although not known to play an essential role in radiation-belt dynamics, drift waves represent a potentially significant mechanism for extracting free energy from magnetospheric particle distributions by causing diffusion across field lines.

III.6 Bohm Diffusion

Electric Drift Velocity. In the absence of collisions and wave-particle interactions, the response of a charged particle to an electric field \( E_\perp \) imposed across \( B \) is the drift given by (1.53) or (3.06). For a simple derivation of this fact, consider that the transverse (to \( B \)) electric field vanishes in a Lorentz frame moving at velocity \( v_0 \) such that

\[
c E_\perp + v_0 \times B = 0.
\]

If \( B \) is uniform, a particle can only execute gyration in this frame and cannot execute translational motion along \( B \). The cross product between \( B \) and (3.31) then yields (1.53) or (3.06) as the velocity of the Lorentz transformation, i.e., of the guiding-center motion across \( B \). If \( B \) is not uniform, then there are additional guiding-center forces equivalent to \( qE \). Replacement of \( qE \) in (1.53) or (3.06) by the sum of all forces \( F \) acting on a particle yields a drift velocity

\[
v_d = \frac{c}{qB^2} F \times B.
\]

The validity of (1.53) or (3.06) requires only that \( r < c \). Guiding center forces requiring an average over gyration, e.g., the forces \(- (M/q)NB \) and \(- (p^2/m)\hat{\mathbf{B}}/\partial \theta \) leading to gradient and curvature drifts (see Section 1.5), limit (3.32) to drift velocities much less than \( c \Omega_1 L_\perp \) in absolute value. Since the gradient-curvature drift velocity is in fact of order \( c^2 \Omega_1 L_\perp \), this means only that the general validity of (3.12) is limited to \( |q| \ll 1 \), as previously assumed.

Effective Collision Frequency. In causing diffusion with respect to energy, pitch-angle, and \( L \) value, wave-particle interactions have an effect quite analogous to that of interparticle collisions. For this reason it is often

\(^{24}\)Drift waves can be destabilized under a variety of conditions [60]. The present calculation illustrates only one example.

convenient to think in terms of an effective collision frequency \( 1/\tau_\perp \) to which the various diffusion coefficients can be related, just as if interparticle collisions were the agent responsible for the diffusion. This equivalent collision frequency is said to produce anomalous transport, in the sense that the diffusion exceeds that which would result from Coulomb collisions alone. Thus, the quantity \( 1/\tau_\perp \) generally exceeds the Coulomb collision frequency.

The mean (phase-averaged) force exerted by collisions and wave-particle interactions can be represented by \(- (m/\tau_\perp) v_\perp \). If \( B \) is uniform, then the net drift velocity resulting from the imposition of \( E_\perp \) across \( B \) is given by

\[
v_d = \frac{c}{L} E_\perp \times \hat{\mathbf{B}} + \left(\frac{1}{\tau_\perp} v_\perp \right) v_\perp \times \hat{\mathbf{B}} - \hat{\mathbf{E}}_\perp,
\]

where \( \Omega_1 = \frac{-q}{L_\perp} \). This result indicates a Hall mobility

\[
\mu_H = \frac{(c/\Omega_1^2)}{(\Omega_1^2 \tau_\perp^2) \left[ 1 + (\Omega_1 \tau_\perp)^2 \right]^{-1}}
\]

in the direction of \( \hat{\mathbf{E}}_\perp \) and a Pedersen mobility

\[
\mu_P = -\frac{(c/\Omega_1 \tau_\perp)}{(1 + (\Omega_1 \tau_\perp)^2)^{-1}}
\]

in the direction of \( \hat{\mathbf{E}}_\perp \). The Pedersen mobility approaches zero in the limit of no "collisions" (\( \Omega_1^2 \tau_\perp^2 \gg 1 \)) and approaches \( q_\perp/m \) in the limit of "collision" dominance (\( \Omega_1^2 \tau_\perp^2 \ll 1 \)). The maximum absolute value (\( c/2 \Omega_1 \)) of \( \mu_P \) is attained when \( \Omega_1^2 \tau_\perp^2 = 1 \).

Diffusion Coefficient. The purpose of calculating the Pedersen mobility is ultimately to obtain the diffusion coefficient related to it, i.e., the coefficient for the stochastic transport of particles across adiabatic drift shells. The connection between mobility and diffusion is given [61] by

\[
D_{LL} = (p_{\perp}/2m_a) \mu_\perp.
\]

Since \( L \) is a dimensionless scale variable defined by the earth radius \( a \), the quantity \( D_{LL} \) must be interpreted as \( a^2 D_{LL} \). It follows that

\[
D_{LL} = (p_{\perp}/ma^2)^2 \left[ 1 + (\Omega_1 \tau_\perp)^2 \right]^{-1}
\]

If \( \tau_\perp \) is now considered an adjustable parameter, the magnitude of \( D_{LL} \) can be maximized by setting \( \tau_\perp = \Omega_1^{-1} \). In other words, there is an upper bound, given by

\[
D_{LL} = (p_{\perp}/2ma^2 \Omega_1)^2 \Omega_1,
\]

on the coefficient of radial diffusion. No adjustment of \( \tau_\perp \) can produce a value of \( D_{LL} \) larger than \( D_{LL} \). A process in which \( D_{LL} \sim D_{LL} \) is characterized as Bohm diffusion [62]. It represents the most expedient means
available to a hot plasma for erasing an unstable spatial gradient (cf. Section III.5) in the distribution function, and in this sense is analogous to strong pitch-angle diffusion (Section II.7), which has the same property relative to unstable gradients of $\mathbf{f}$ in momentum space.

There is, however, no reason why Bohm diffusion must cause strong pitch-angle diffusion. As in Section III.5, the "collisions" could easily act preferentially in the direction normal to $\mathbf{B}$, an option not available to interparticle collisions. Thus, the anomalous Ohmic mobility $\mu_0 = (v \cdot \mathbf{B})(\mathbf{E} \cdot \mathbf{B})$ is given by $q_0 t/m$, where $t_0$ may be entirely different in magnitude from $t$ in (3.34). In the event that $t_0 > t$, there may be very few particles scattered into the loss cone in the course of Bohm diffusion. Conversely, strong diffusion requires only that $\Omega_2 t_0 \ll 1$ and $\Omega_2 t \ll 1$. These conditions do not necessarily imply $|\Omega_1| t_0 \ll 1$, as required for Bohm diffusion.

An examination of (3.6) indicates that $D_{LL} \approx L^2 |\Omega_1|$. No radiation-belt observations are known to require nearly this large a value of $D_{LL}$, but the storm-time ring current occasionally appears to exhibit Bohm diffusion in the vicinity of the plasmapause. The plasmasphere tends to destabilize the ring current against electromagnetic ion-cyclotron turbulence (see Section II.6) by drastically reducing the minimum resonant energy given by (2.69b). Bohm diffusion is sometimes invoked [63], in addition to the adiabatic gradient-curvature drift, as a means of transporting ring-current protons into the plasmasphere from the exterior region in which $N_e$ is very small ($\sim 0.1$ cm$^{-3}$ during a magnetic storm). Even if the presence of strong pitch-angle diffusion, which the resulting ion-cyclotron turbulence causes, the Bohm diffusion coefficient would transport ring-current protons ($e \sim 10^{-3}$) a root-mean-square distance $\sim 0.5 a$ relative to the plasmapause during the lifetime $1/2$ given by (2.77).

III.7 Shell Splitting

As described in Section I.7, drift-shell splitting is a purely adiabatic phenomenon that violates none of the invariants. Radial diffusion, by definition, violates the third invariant. Pitch-angle diffusion violates either or both of the first two invariants, usually both. In a symmetrical magnetosphere, the incidentally associated radial diffusion coefficient $D_{LL} \sim e^2 L^2 |\Omega_1|$ would be too small to be of significance for radiation-belt particles ($e^2 \ll 1$). In the presence of azimuthal asymmetry and shell splitting, however, pitch-angle diffusion automatically produces an additional violation of the third invariant. The shell-tracing results obtained in Section I.7 permit this effect to be evaluated for arbitrary values of the equatorial pitch angle.

The basic equation governing the process under consideration is (cf. Section II.2)

$$D_{LL} = \langle \partial L/\partial x \rangle^2 D_{ax} = \langle x/\bar{y} \rangle^2 |\partial L/\partial y|^2 D_{ax}.$$

Thus, if the values of $L = 2 \pi a^2 B_0 |\Phi|^{-1}$, among identical particles having mirror points on a common field line, vary with equatorial pitch angle, then pitch-angle diffusion of these particles on this field line automatically produces diffusion with respect to $L$ [5]. The partial derivatives are evaluated by holding constant the quantity $D_{ax}$, typically the particle energy or first invariant. The drift average denoted by the angle brackets necessarily yields a positive-definite $D_{LL}$ [65].

External Multipole. In the case of magnetic shell splitting, as summarized by (3.11), pitch-angle diffusion leaves $L_d$ and $\Phi$ invariant at the scattering site, and so the quantity $\partial L/\partial y$ is given by

$$\partial L/\partial y = - (B_2/2 \pi a)(a/b)^4 L_d^2 |D(y)|^{-2} \times \left[ Q'(y) D(y) - D' y Q(y) \right].$$

If pitch-angle diffusion is distributed uniformly with respect to longitude, i.e., if $D_{ax}$ is independent of $\phi$, then to lowest order in $e^2 = (B_2/B_0)(a/b)^4$ it follows that

$$D_{LL} = (e^2/2 \pi)^2 (B_2/2 \pi a)(a/b)^4 |L_d^0 |^2 |D(y)|^{-4} \times \left[ Q'(y) D(y) - D' y Q(y) \right]^{-2} D_{ax}.$$

With the aid of (1.36) and (1.79), the function $(x^2/18) e^2 (6D(y))^{-4} \times \left[ Q'(y) D(y) - D' y Q(y) \right]^{-2}$, which expresses the pitch-angle dependence of $D_{LL}/D_{ax}$, has been plotted in Figure 28 [66]. This function reduces to $25x^2/18$ in the limit $x^2 = 1 - y^2 \ll 1$, in which case (3.39) reduces to the expression

$$D_{LL} \approx (x^2/18)(5 B_2/B_0)(a/b)^4 L_d^0 D_{ax}$$

$$\approx 0.61 x^2(a/b)^4 L_d^0 D_{ax}.$$

As an upper bound on radial diffusion induced by magnetic shell splitting, this expression remains valid for $|x| \leq 0.9997$; it fails only deep within the loss cone. As an approximate expression for $D_{LL}$, equation (3.40) remains valid within a factor of two only for $|x| \leq 0.6$, while (3.39) is correct to within a few percent for all $x$ when evaluated via (1.36) and (1.79).

The complete Jacobian entering (2.12), when pitch-angle diffusion violates $\Phi$, depends upon the nature of the quantity conserved in the
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Fig. 28. Relation between $D_{LL}$ and $D_{xx}$ for shell splitting caused by noon-midnight magnetic asymmetry [66], as given by harmonic-bounce approximation (dashed curve) and by improved approximation (solid curve) based on (1.26) and (1.79).

process. If the conserved quantity is particle energy, then the relevant Jacobian is

$$G(M,J,|\theta|;E,x,L) = -8\pi\nu mL^2a^3xT(y), \quad (3.41a)$$

as deduced from (2.14) and (1.37). If, as in the case of bounce resonance, the conserved quantity is $M$, then the relevant Jacobian is

$$G(M,J,|\theta|;M,x,L) = -8\pi B_0a^3(p/y^2L)xT(y), \quad (3.41b)$$

which follows from (2.27), (1.37), and the fact that $(\partial w/\partial x)_{M,L} = 2MB_0x/L^2y^4$ [cf. (2.33) and (2.34)].

In (3.41a), the distribution function $f$ is considered to depend on $E$, $x$, and $L$. Since $E$ is conserved by the process, the distribution function satisfies [65]

$$\frac{\partial f}{\partial \xi} = \frac{1}{E} \frac{\partial}{\partial L} \left[ \frac{\partial}{\partial L} D_{LL} \frac{\partial f}{\partial L} \right] + \frac{1}{xT(y)} \frac{\partial}{\partial x} \left[ xT(y)D_{xx} \frac{\partial f}{\partial x} \right], \quad (3.42)$$

where $D_{LL}$ is given by (3.39). This equation contrasts strikingly with (3.01), which applies to processes that conserve $M$ and $J$.

Electric Shell Splitting. In the case of electric shell splitting, caused by superposition of (1.52) upon the dipole field (1.16), the relation between $L$ and $y$ at constant $L_d$ and $\varphi$ is expressed by (3.15), provided that

$$W > |qV|$$

around the entire drift shell. The connection between $D_{LL}$ and a $\varphi$-independent $D_{xx}$ is then given by

$$D_{LL} \approx (qE_aL^2/W)(m_0c^2 + W)^2(2m_0c^2 + W)^{-2} \left[ 6D(y) \right]^{-4} \times \left[ Y'(y) T(y) - T'(y) Y(y) \right]^2 (x^2/2y^2) D_{xx} \quad (3.43)$$

for pitch-angle diffusion at constant particle energy $W$. With the aid of (1.28), (1.31), and (1.36), the function $(x^2/8y^2) \left[ 6D(y) \right]^{-4} \times \left[ Y'(y) T(y) - T'(y) Y(y) \right]^2$ has been plotted in Fig. 29. This function indicates the pitch-angle dependence of $D_{LL}/D_{xx}$ in the presence of electric shell splitting, and approaches $x^2/162$ in the limit $x^2 \ll 1$. The nonrelativistic limit $(W \ll m_0c^2)$ of (3.43) therefore reads

$$D_{LL} \approx (x^2/162)(qE_aL^2/W)^2D_{xx} \quad (3.44)$$

for $x^2 \ll 1$, and represents a serious (factor-of-two) underestimate for $D_{LL}$ only if $|x| \gtrsim 0.6$.

Fig. 29. Relation between $D_{LL}$ and $D_{xx}$ for shell splitting caused by dawn-dusk asymmetry of electrostatic potential [66], as given by harmonic-bounce approximation (dashed curve) and by improved approximation (solid curve) based on (1.28), (1.31), and (1.36).
the ring-current particles. In the true radiation belts, magnetic shell splitting effects exceed those of electric shell splitting.

At particle energies below those typical of the ring current, it is necessary to reconsider the shell-splitting problem in terms of \( (1.65) \). Beyond the plasmapause, such drift shells do not close within the magnetosphere, and the corresponding third invariants are undefined (cf. Fig. 12, Section 1.6). Within the plasmasphere, all shell splitting disappears in the cold-plasma limit, since "zero-energy" particles drift on field-aligned surfaces of constant electrostatic potential.

**Internal Multipoles.** At very low \( L \) values, certain internal geomagnetic multipoles associated with true field anomalies, may cause significant shell splitting among inner-belt particles \([67]\). If electric fields are negligible, the existence of magnetic shell splitting in general can be demonstrated (cf. Section 1.7) by showing that \( \phi B/\phi s^2 \), varies with \( \phi \) around a path of constant \( B \), on the equatorial \( \phi B/\phi s=0 \) surface. This criterion follows from (1.26) and (1.32a), in the sense that the drift shell (which conserves \( M \) and \( J \) ) must depart from the constant-B trajectory for \( x \approx 0 \) if \( \Omega_2 \) varies with \( \phi \); to lowest order in \( x \), the bounce frequency is given by \( \Omega_B = (M/M_B^2 B/\phi s^2) \).

In a dipole field, the value of \( \phi B/\phi s \) is given by \( 9 B_0 a^2 L_\perp^2 \) where \( L_\perp = (B_0/B) \) at \( y = 1 \) [cf. (1.38)]. It proves convenient to display the azimuthal variation of \( \phi B/\phi s^2 \) at constant \( B \) and \( L_\perp \) by plotting \( L_\perp^2 (L_\perp^2 a^2 B_0) \langle \phi B/\phi s \rangle - 1 \) against geometric longitude. This is done\(^{26a} \) in Fig. 30 for \( L_\perp=1 \). The sinusoidal asymptote, approximated by the curve \( L_\perp=\infty \), results from an internal octupole. The octupole corresponds to \( n = -5 \) in (1.46) and the dipole to \( n = -3 \), hence the factor \( L_\perp^2 \) in Fig. 30. Higher multipoles produce the broad South American anomaly (longitude \( 30^\circ - 120^\circ \)) and the narrow South American anomaly (longitude \( 0^\circ - 30^\circ \)). The latter disappears between \( L_\perp=1 \) and \( L_\perp=2 \). The fact that the dipole is off center by \( \approx 0.07a \) toward longitude \( 217^\circ \), necessarily contributes nothing to shell splitting as this is not a field asymmetry. Components of the geomagnetic quadrupole that survive the transformation to off-dipole coordinates can only warp the equatorial \( \phi B/\phi s=0 \) surface as a lowest-order effect. Their second-order (shell-splitting) effects are not discernible in Fig. 30 [67].

\(^{25}\) However, the demarcation between ring-current and radiation-belt parameters is somewhat arbitrary (Fig. 10, Section 1.7).

\(^{26}\) The shell \( L_\perp=1 \) is unphysical in the sense that it intersects the earth's surface (see Section II.1).

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**III.8 Diffusion in More Than One Mode**

For each diffusion mechanism considered in Chapters II and III, the diffusion tensor \( D_{ij} \) (see Section II.1) can be diagonalized by a proper choice of variables, i.e., by transforming from the coordinates \((M, J, \phi)\)
to an equivalent set of functionally independent variables. Mixed partial derivatives in (2.12) are thus eliminated. Vanishing eigenvalues (diagonal elements, \(i = j\)) of the transformed diffusion tensor \(D_{ij}\) correspond to conservation laws of the diffusion mechanism \([68]\). For example, pure pitch-angle diffusion, \(i.e.,\) diffusion at constant particle energy, corresponds to \(D_{M J} = 0\) and (in the absence of shell splitting) \(D_{LJ} = 0\). Pure third-invariant diffusion (Sections III.1–III.3) has the property that \(D_{LMM} = D_{LJ} = 0\). A summary of various diffusion mechanisms, their conservation laws, and the Jacobians of their respective diagonalizing transformations is given in Table 8.

Table 8. Diffusion Variables and Associated Jacobians

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<th>Interaction</th>
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<td>Elastic Collisions</td>
<td>(E, (\Phi))</td>
<td>(G(M, J, \Phi, E, x, L))</td>
</tr>
<tr>
<td>(without recoil)</td>
<td>(E, (\Phi))</td>
<td>(=8\pi^2 p^2 L^2 a^3 x T(y))</td>
</tr>
<tr>
<td>Cyclotron Resonance</td>
<td>(E, (\Phi))</td>
<td>(G(M, J, \Phi, E, x, L))</td>
</tr>
<tr>
<td>Bounce Resonance</td>
<td>(M, (\Phi))</td>
<td>(G(M, J, \Phi, M, x, L))</td>
</tr>
<tr>
<td>Dipole Resonance</td>
<td>(M, K)</td>
<td>(=8\pi^2 a^3 (2m_B M/L^2)^{1/2} x T(y))</td>
</tr>
<tr>
<td>Drift Resonance</td>
<td>(M, K)</td>
<td>(G(M, J, \Phi, M, K, L))</td>
</tr>
<tr>
<td>Bimodal Diffusion</td>
<td>(L)</td>
<td>(G(M, J, \Phi, x, L))</td>
</tr>
</tbody>
</table>

(Parenthesized "invariant" quantities are either approximately or conditionally conserved).

No special difficulty of concept arises when two or more diffusion mechanisms act simultaneously. If the concurrent processes satisfy the same conservation laws, then a single transformation of variables will suffice to make the diffusion tensor diagonal. If not, \(i.e.,\) if the conservation laws for kinematical variables are not common to the various diffusion mechanisms acting concurrently, then the problem is said to involve more than one mode of diffusion. In this case, the diffusion equation is at least two-dimensional with respect to the kinematical variables. This property presents no special difficulty, since two-dimensional diffusion equations, \(e.g.,\) (3.42), have already appeared in the context of unimodal diffusion. In constructing a bimodal diffusion equation, however, it is essential to evaluate the partial derivatives in accordance with the conservation laws of the respective modes. For example, if radial diffusion at constant \(M\) and \(J\) (coefficient \(D_{LJ}\)) is superimposed upon pitch-angle diffusion at constant \(E\) (coefficient \(D_{LJ}\)) in the presence of magnetic shell splitting, the equation governing this bimodal process is

\[
\frac{\partial f}{\partial t} = E\frac{\partial}{\partial L} \left[ \frac{1}{2} D_{LL} \frac{\partial f}{\partial L} \right] + \frac{1}{2} \frac{\partial}{\partial x} \left[ x T(y) D_{xJ} \frac{\partial f}{\partial x} \right] + \frac{1}{2} \frac{\partial}{\partial L} \left[ \frac{x^2 L^2}{12} [Q(y)D_{LL} - R(y)Q(y)]^2 D_{xJ} \frac{\partial f}{\partial L} \right] \tag{3.46}
\]

a result obtained by consolidating (3.01), (3.39), and (3.42).

The right-hand side of (3.46) has the form of minus the "divergence" of a diffusion current for each mode (cf. Sections II.1 and II.2). The radial (trans-L) component of the diffusion current has the form 

\(-D_{LL}(\partial f/\partial L)_{L,I,J} \) for the sudden-impulse mode and the form 

\(-\langle x/y \rangle^2 \partial (\partial L/\partial y)^2 D_{xJ} \) in the shell-splitting mode (cf. (3.37), Section III.1). For outer-belt electrons at \(L \geq 5\), it is interesting that \((\partial f/\partial L)_{L,I,J}\) is typically positive, while \((\partial f/\partial L)_{L,J}\) is typically negative (see Fig. 1 and Section IV.6). The diffusion current across \(L\) thus consists of an inward part conserving \(M\) and \(J\), which tends to energize the diffusing particles, and an outward part conserving \(E\). The net result is that, for particles diffusing "bimodally" from an external source into the outer belt, the gain in energy typically exceeds that predicted on the basis of constant \(M\) and \(J\) (see Section III.1).

Even if shell-splitting effects are neglected, \(e.g.,\) by taking \(B_2 = 0\), the diffusion equation (3.46) is two-dimensional in the sense that no overall conservation law relates \(x\) and \(L\). Thus, an individual particle from the distribution \(f(E, x, L, t)\) may random-walk a complete cycle in \(x\) and \(L\), as illustrated in Fig. 31. In the absence of shell splitting,
III. Radial Diffusion

the radial diffusion in (3.46) occurs at constant M and J. According to (1.34b), the variation of particle energy with L is governed by the relationship

\[ \frac{\partial \ln p}{\partial \ln L_{M,J}} = -3 \frac{[D(y)/T(y)]}{L}. \]  

(3.47)

It follows that a clockwise cycle in Fig. 31 (inward diffusion at generally smaller y than outward diffusion) represents a net loss in particle energy, while a counter-clockwise cycle causes a particle to gain energy. In this context, bimodal diffusion acts as a "thermalization" mechanism, whereby an initially narrow energy spectrum of particles can become distributed to both higher and lower energies than pure conservation of M and J would allow [69].

Reduced Diffusion Equations. For many problems involving radiation-belt diffusion, it is considered appropriate to simplify (3.46) by means of approximations that reduce the diffusion equation to one spatial dimension. Simplifying approximations of this type are often indicated when the observational data are not sufficiently complete to impose meaningful boundary conditions on (3.46). In many cases the observations cover too limited a range of parameter space to make full use of (3.46). Reduction of the diffusion equation to one dimension, however justified, does require that bimodal cycles of the type illustrated in Fig. 31 be neglected. This is part of the cost of analytical simplification.

A naive means of reducing (3.46) is to neglect shell-splitting effects and to replace the pitch-angle diffusion term by a simple loss term of the form \(-f \tau\). In this approximation [70] the diffusion equation reads \([\text{cf. } (2.09)]\)

\[ \frac{\partial \mathcal{J}}{\partial t} = \frac{\partial}{\partial L} \left[ \frac{1}{E} \frac{\partial \mathcal{J}}{\partial L} \right]_{L_{M,J}} \frac{\mathcal{J}}{\tau} \]  

(3.48)

and applies to \(f(M,L)\) at \(J=0\). The pitch angles of particles having in common their values of \(M/\alpha^2\) and \(L\) are mixed thoroughly on a time scale \(\sim \tau/5\) (see Section II.7). The representation of pitch-angle diffusion as a simple loss term, as in (3.48), essentially requires that \(\sigma_0/\tau\) greatly exceed \(\sigma E/\partial t\) in absolute value [23]. The diffusion coefficient \(D_{LL}\) is then interpreted as an average over particles sharing the same values of \(M/\alpha^2\) and \(L\), respectively.

A more sophisticated view of the reduction described in the paragraph above is that a new variable \(\xi = M/\alpha^2\) has been introduced, and that \(\xi\) is approximately conserved by both \(D_{LL}\) and \(D_{\alpha x}\) [71]. From this viewpoint, the form of (3.48) should be governed by the Jacobian [5]

\[ G(M,J,M; \xi, x, L) = -(8 \pi a^2 B_{\alpha}/E_0^3) x T(y)(2m_e B_{\alpha} L)^{1/2}, \]  

(3.49)

which has been included in Table 8. With this Jacobian, the reduced (to one dimension) diffusion equation evidently has the form

\[ \frac{\partial \mathcal{J}}{\partial t} = \frac{\partial}{\partial L} \left[ \frac{1}{E} \frac{\partial \mathcal{J}}{\partial L} \right]_{L_{M,J}} \frac{\mathcal{J}}{\tau}. \]  

(3.50)

The practical discrepancy between (3.48) and (3.50) is slight, amounting only to a square root of \(L\) in the metric. Since \(D_{LL}\) typically varies as \(L^{1/2}\) (see Sections III.2 and III.3) for radiation-belt particles, it is difficult to imagine that seriously different geophysical predictions might emerge from (3.48) and (3.50), although (3.50) is perhaps preferable in terms of self-consistency.

In either representation the transport coefficients may certainly vary with \(L\) and perhaps also with \(\xi \) \((=M\) at \(J=0\)) and/or time. Since \(D_{LL}\) and \(\tau\) arise from operations on the entire pitch-angle distribution, it would be meaningless to give either a dependence on \(x\) or \(y\). This degree of freedom has been sacrificed in reducing (3.46) to one dimension. The conservation of \(\xi\) is clearly an idealization that breaks down for \(x \sim 1\), but the presence of a loss cone (see Section II.7) assures that \(f\) is small there [23]. Thus, the effective radial-diffusion coefficient \(D_{LL}\) is heavily weighted by the behavior of particles for which \(x^2 \ll 1\), i.e., for which radial diffusion at constant \(M\) and \(J\) very nearly conserves \(\xi\).

If the time scale for pitch-angle mixing \((\sim \tau/5)\) is comparable to that for radial diffusion, then a simplified equation such as (3.50), which assigns to \(f\) the lowest mode of pitch-angle diffusion (see Section II.7), cannot apply unless \(D_{LL}\) is substantially independent of \(x\) (cf. Sections III.2 and III.3). Thus, radial diffusion caused by electrostatic impulses may lend itself to analysis via (3.50), but that caused by magnetic impulses will ordinarily bias \(f\) toward higher modes of pitch-angle diffusion. In this case a more general treatment is required.

If the need to circumvent (3.46) is compelling, it may be possible to expand \(f(\xi, x, L; t)\) in pitch-angle eigenfunctions \(g_n(\xi)\) that are even in \(x\) (even parity required because of homogeneity over bounce phase). An expansion [71] of the form

\[ f(\xi, x, L; t) = \sum_n g_n(\xi) g_n(x) \]  

(3.51)

This requirement is often overlooked in the interest of expedience.
III. Radial Diffusion

with the boundary condition \( g_\alpha(x) = 0 \) is justified if \( x \) is independent of \( L \), and the \( L \) dependence of \( D_{xx} \), which is factorable, i.e., if \( D_{xx} \) is the product of a function of \( L \), \( \zeta \), and \( t \) times a function of \( x \). These conditions on \( x \) and \( D_{xx} \) are probably well satisfied in the outer zone. It is convenient to assume further that \( D_{xx} \) and \( D_{L} \) are time-independent. In this case the approximate diffusion equation \([\text{cf. } (3.50), (3.49), \text{ and } (3.46)]\) with \( B_2 = 0 \) (i.e., without shell splittings)

\[
\frac{\partial \tilde{T}}{\partial t} = \left[ \frac{L^{5/2} D_{xx} \partial^2 \tilde{T}}{\partial \xi^2} + \frac{1}{x T(y)} \frac{\partial}{\partial x} \left[ x T(y) D_{xx} \frac{\partial \tilde{T}}{\partial x} \right] \right]_{\xi} \tag{3.52a}
\]

can be simplified by virtue of the eigenvalue property

\[
\frac{1}{x T(y)} \frac{\partial}{\partial x} \left[ x T(y) D_{xx} g_\alpha(x) \right] = -\lambda_\alpha(L) g_\alpha(x), \tag{3.52b}
\]

where \( \lambda_\alpha(L) \) is the decay rate characteristic of the pitch-angle eigenmode \( g_\alpha \) (cf. Section II.7).

The normalized eigenfunctions corresponding to distinct eigenvalues \( \lambda_\alpha \) are orthogonal in the sense that

\[
\int_0^x x T(y) g_\alpha(x) g_\beta(x) dx = \delta_{\alpha \beta}. \tag{3.53}
\]

Application of (3.52) to (3.51) therefore implies that

\[
\frac{\partial a_m}{\partial t} = L^{5/2} \frac{\partial}{\partial \xi} \left[ \frac{L^{5/2}}{x T(y)} \sum \overline{D}_{LL} \frac{\partial a_n}{\partial \xi} \right] - \lambda_m a_m, \tag{3.54a}
\]

where

\[
\overline{D}_{LL} = \int_0^x x T(y) D_{LL} g_m(x) g_n(x) dx. \tag{3.54b}
\]

If \( D_{LL} \) is independent of \( x \), as is approximately true in radial diffusion caused by electrostatic impulses (see Section III.3), then the matrix \( \overline{D}_{LL} \) is diagonal in the sense that \( \overline{D}_{LL} = D_{LL} \delta_{mn} \). In this case the functions \( a_m(\zeta, L; t) \) and \( a_n(\zeta, L; t) \) in (3.54a) are decoupled for \( m \neq n \) and diffuse separately with respect to \( L \) \([71]\). If \( \tilde{T}(\zeta, x, L; t) \) is initially in its lowest pitch-angle eigenmode \( g_0(x) \), therefore, it will continue in this eigenmode and diffuse according to (3.50) as time goes on. On the other hand, off-diagonal elements of \( \overline{D}_{LL} \) which are obviously substantial in radial diffusion caused by magnetic impulses (see Section III.2), serve to couple distinct pitch-angle eigenmodes and thereby "excite" modes not present in the initial configuration of \( \tilde{T}(\zeta, x, L; t) \).

III.8 Diffusion in More Than One Mode

Inner-Zone Protons. For particles that do not undergo significant pitch-angle diffusion, the fundamental radial-diffusion equation is (3.01). Very energetic \( (E \geq 100 \text{ MeV}) \) inner-zone protons are believed to be in this category. The principal source for these particles is known as CRAND (see Section III.1): cosmic rays incident on the upper atmosphere eject high-energy neutrons that beta-decay with a mean life \( \tau_\beta \) (\( \sim 10^3 \text{ sec} \)) in their own rest frame. At low latitudes the vertical flux \( J_v \) of these "albedo" neutrons is believed to be given \([72]\) by

\[
J_v \approx 0.044 (E/1 \text{ MeV})^{-1.86} \text{cm}^{-2} \text{sec}^{-1} \text{MeV}^{-1} \tag{3.55}
\]

at the top of the atmosphere \( (r = a + h, \text{ cf. Sections II.2 and II.7}) \). The presence of these decaying neutrons \[39\] requires that a proton source term \[38\]

\[
S \approx (J_v/(2\pi \tau_B \phi))^2 \tag{3.56a}
\]

be added to the right-hand side of (3.01). The geometric injection coefficient \( \phi \) for equatorially mirroring protons is estimated by the expression \[73\]

\[
\phi \approx (2/\pi)(a + h)/L_0. \tag{3.56b}
\]

The arc sine represents the half angle subtended by the earth's atmosphere at the site of proton injection (neutron decay) in a model centered-dipole field \[39\].

The inner-zone protons injected by CRAND lose energy to free and bound ionospheric electrons \([\text{cf. } (2.04)]\) but gain energy from the secular decrease of \( B_0 \) \([\text{cf. } (2.05)]\). Both processes leave the equatorial pitch angle invariant. The energy gain is an adiabatic effect, and so is automatically included if the problem is posed in the invariant coordinates \( M, J, \phi \), i.e., in the form that reduces to

\[
\frac{\partial \tilde{T}}{\partial t} = S + \frac{\partial}{\partial \phi} \left[ D_{\phi \phi} \frac{\partial \tilde{T}}{\partial \phi} \right] + \left( \frac{4\pi \gamma^2 m_e}{m_r} \right) \left[ \frac{\partial}{\partial M} \frac{\partial C}{\partial M} \right]_{M, \phi}. \tag{3.57a}
\]

\[29\]The mean free path of a 100-MeV neutron before beta decay is of the order of one astronomical unit. Decay within the magnetosphere therefore does not significantly deplete the flux of cosmic-ray-albedo neutrons.

\[30\]For this derivation of \( S \), it is assumed that the neutron flux is isotropic at the top of the atmosphere, so that the omnidirectional neutron flux \( J_\theta \) is twice the vertical flux \( J_v \). The unidirectional neutron flux above the atmosphere remains \( (1/2\pi)J_\theta \), by Liouville's theorem (Section I.3), for gyrophase angles compatible with ejection from the atmosphere. The result is a gyrophase-averaged proton source \((dJ_p/dt)_u = (1/2\pi)J_\phi, i.e., \text{ a source for } f(=J_\phi/p^2) \) given by (3.56).
The drift shell corresponding to given values of \( \gamma \) may also contribute a significant secular variation having similar consequences (see Section II.2).

\[
C = N_{\gamma}[(\gamma^2 - 1)^{1/2} \ln \left( \frac{\Delta E}{m_e c^2} \right) + \sum_{i} N_{i} Z_{i} (\gamma^2 - 1)^{1/2} \ln \left( \frac{2 m_i c^2 (\gamma^2 - 1)}{l_i} \right)]
\]  
(3.57b)

The mirror field \( B_{\phi} \) is given in terms of the invariant coordinate \( \phi \) by \( B_{\phi} = (1 / n^2 \phi^2 B_{\phi} \phi^2 \phi^4) \), and thus contains an explicit time dependence (that of \( B_{\phi} \)). Expressed as functions of \( K^2 = (J^2 / 8 m_0 M) \) and \( \phi \), the drift-averaged atmospheric densities \( N_{\gamma} \) also vary with time. The drift shell corresponding to given values of \( K \) and \( \phi \) not only contracts temporally (since \( B_{\phi} / B_{\phi} < 0 \)), but also moves laterally relative to the earth so as to remain concentric with the dipole axis \(^{31}\) (apart from the effects of magnetic anomalies, cf. Section II.1). A growing dipole-offset distance imparts an additional increase to \( N_{\gamma} \) with time for atmospheric constituents whose densities decrease with altitude.

The secular variation \(^{32}\) of \( B_{\phi} \) on a time scale \( \sim 2000 \text{yr} \) prevents (3.57) from having a steady-state \( (\partial f / \partial t = 0) \) solution with which the inner proton belt can be identified. Thus, the present state of protons in the inner zone is the result of a long and continuing process of evolution. According to Fig. 14 (Section II.2) protons presently trapped in the inner zone may well have resided there for the past thousand years or more. An integration of (3.57) over this geomagnetic history may be fraught with uncertainty, in view of the available observations. Such a treatment appears to be necessary, however.

In much of the inner zone, the secular decrease of \( B_{\phi} \) energizes trapped protons more efficiently than does inward radial diffusion at constant \( M \) and \( J \). Typical time scales for the latter process at \( J = 0 \) have been indicated by broken lines in Fig. 14 (Section II.2). For this purpose, the diffusion “current” \(-D_{\phi \beta}(\partial f / \partial L)_{M,J} \) identified following (3.46) has been utilized to construct an effective “velocity”

\[
L = -D_{\phi \beta}(\partial \ln f / \partial L)_{M,J}
\]  
(3.58)

Insertion of \( 10 / L \) as a likely upper bound \(^{38, 39}\) for \( (\partial \ln f / \partial L)_{M,J} \) leads to the estimate \([\text{cf. (2.05)}]\) that

\[
\frac{1}{E} \frac{dE}{dt} = \frac{E \gamma + 1}{B_\phi} \frac{\partial \phi}{\partial L} \frac{1}{2 \gamma L} \lesssim \frac{30}{D_{\phi \beta}} \left( \frac{\gamma + 1}{2 \gamma} \right) D_{\phi \beta}
\]  
(3.59)

\(^{31}\)At present the distance between the dipole axis and the geocenter is growing at a rate \( \sim 2 \text{km/yr} \).

\(^{32}\)Other axially symmetric internal multipoles \((2^n)\) of odd-\(n\) order \((e.g., \text{octupole})\) may also contribute a significant secular variation having similar consequences \([36]\).