“CLASSICAL” GRAVITATIONAL DYNAMICS

This topic isn’t usually part of “Physics III,” but: (1) it’s an important enough part of astronomy that we ought to spend some time on it, and (2) the story leads into Einstein’s relativity, which comes next.

Most of this topic will involve the two-body problem, i.e., a complete description of the dynamics of two massive “particles” that feel each other’s gravitational attraction, and that fly around in a vacuum (i.e., ignoring any “external” effects).

Once the two-body problem is understood, one can go on to study things like:

- **The three-body problem:** formally unsolvable, but interesting in certain limits (like a tiny body flying around in the gravitational field of a binary system).

- **N-body dynamics** (for $N \gg 1$): e.g., stars in a galaxy, galaxies in a cluster, and so on.

- **Gravitational hydrodynamics:** similar to $N \gg 1$, but the “bodies” merge together into a continuous self-gravitating fluid.

You’ve seen the historical overview. Copernicus and Galileo caught the heliocentric bug, Tycho Brahe took better data than the ancients, and his student Kepler eventually made sense of the data in terms of **empirical** mathematical “laws:”

**Johannes Kepler** (1571-1630)

1. The orbit of a planet is an ellipse with the Sun at one of the two foci.
2. A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
3. The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.
Then along came Newton and his apple (and his calculus) to formulate more general laws of physics that describe *both* terrestrial & celestial motion:

*Isaac Newton* (1643-1727)

1. A body will remain at rest, or moving at a constant velocity, unless it is acted on by an unbalanced force $F$.
2. $F = dp/dt = ma$.
3. When body 1 exerts a force on body 2, body 2 simultaneously exerts an equal & opposite force on body 1: $F_{12} = -F_{21}$.

There is also the unofficial fourth law: Newton’s universal law of gravitation. The force on particle 1, due to particle 2 is:

$$F_{12} = -\frac{Gm_1m_2}{|r_1 - r_2|^2} \hat{r}_{12}$$

where $|r_1 - r_2|$ is the scalar distance between the two particles (which we often just call $r$), and the unit vector that points radially from the source of gravity (particle 2) to the thing that feels it (particle 1) is

$$\hat{r}_{12} = \frac{r_1 - r_2}{|r_1 - r_2|}.$$

Note that this vector definition is independent of coordinate system!

We will start with Newton’s laws as “axioms,” and show how Kepler’s laws arise as a natural consequence of them.

We’ll eventually get down to solving $F = ma$ for the motions of the two bodies, but there are some things to prove first that will make our life much easier down the road:

- Deriving fundamental **conservation laws** from Newton’s laws.
- Choosing a **coordinate system** that simplifies the math.
Conservation Laws

After Kepler & Newton, there was a third key historical step in physics...

Emmy Noether (1882-1935)

Every fundamental symmetry of nature has a corresponding conservation law:

1. A system obeying translation symmetry (i.e., laws are the same at any location) conserves momentum.

2. A system obeying rotational invariance conserves angular momentum.

3. A system obeying time invariance conserves energy.

We won’t derive this, but it’s good to know there’s an additional level of unifying symmetry and order.

So, which conservation laws do we need?

The usual first one is conservation of mass. In our two-body problem, it’s trivially satisfied:

$$\frac{d}{dt} \left( m_1 + m_2 \right) = 0 .$$

Even if we extend it to allow for one body to “donate” some of its own mass to the other (which we won’t!) this still works.

Next is conservation of momentum. For particle $i$, the classical definition of momentum is

$$p_i = m_i v_i$$

and it’s clear that Newton’s 2nd law is more simply described in terms of it:

$$F_i = m_i a_i = m_i \frac{dv_i}{dt} = \frac{dp_i}{dt} .$$

For our two-body system, how does the total momentum ($p_{\text{tot}} = p_1 + p_2$) change in time?

$$\frac{dp_{\text{tot}}}{dt} = \frac{d}{dt} \left( p_1 + p_2 \right) = F_{12} + F_{21} = 0$$

where Newton’s 3rd law is embedded in the form of his universal law of gravitation: i.e., the force on 1 (due to 2) is equal and opposite to the force on 2 (due to 1).
Thus, $\mathbf{p}_{\text{tot}}$ is constant over time, and total momentum is conserved.

Notes about this:

- I’ll probably use $\mathbf{F}_i$ and $\mathbf{F}_{ij}$ interchangeably for the force on particle $i$ (due to $j$).
- Conservation of total momentum is true for $N$-body systems, too. For every particle $i$ being tugged on by $j$, there’s also the equal and opposite force on $j$ by $i$... in the big sum, all pairs of forces cancel out!

Next is conservation of angular momentum. This is another vector quantity that’s defined as

$$\mathbf{L}_i = \mathbf{r}_i \times \mathbf{p}_i$$

and it’s important to note that because $\mathbf{L}_i$ depends on $\mathbf{r}_i$, it is always defined relative to a specific choice of origin.

For a single particle, how does it change in time?

$$\frac{d\mathbf{L}_i}{dt} = \left( \mathbf{r}_i \times \frac{d\mathbf{p}_i}{dt} \right) + \left( \frac{d\mathbf{r}_i}{dt} \times \mathbf{p}_i \right) = (\mathbf{r}_i \times \mathbf{F}_i) + (\mathbf{v}_i \times \mathbf{p}_i) = \mathbf{r}_i \times \mathbf{F}_i .$$

Because $\mathbf{v}$ is parallel to $\mathbf{p}$, the second cross product is zero.

The quantity $\mathbf{r}_i \times \mathbf{F}_i$ is called the torque on particle $i$.

For our two-body system,

$$\frac{d\mathbf{L}_{\text{tot}}}{dt} = (\mathbf{r}_1 \times \mathbf{F}_{12}) + (\mathbf{r}_2 \times \mathbf{F}_{21}) = (\mathbf{r}_1 \times \mathbf{F}_{12}) - (\mathbf{r}_2 \times \mathbf{F}_{12})$$

$$= (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}_{12} .$$

However, the vector $\mathbf{r}_1 - \mathbf{r}_2$ points along a line joining the two particles. So does $\mathbf{F}_{12}$! Thus, this is a cross product of two parallel vectors, and

$$\frac{d\mathbf{L}_{\text{tot}}}{dt} = 0 \quad \text{i.e.,} \quad \mathbf{L}_{\text{tot}} = \text{constant over time}$$

and the total angular momentum is conserved.
Lastly, we’ll examine **conservation of total energy**.

For a single particle, we begin by defining its **kinetic energy** (a scalar). You may have seen it written in two equivalent ways:

\[
K_i = \frac{1}{2} m_i v_i^2 = \frac{p_i^2}{2m_i}
\]

where we use the shorthand \( v_i^2 = |v_i|^2 = (v_i \cdot v_i) \), and so on.

How does it change in time? Let’s use the product rule...

\[
\frac{dK_i}{dt} = \frac{m_i}{2} \frac{d}{dt}(v_i \cdot v_i) = \frac{m_i}{2} \left[ 2 \left( \frac{dv_i}{dt} \cdot v_i \right) \right]
\]

\[
= \frac{dp_i}{dt} \cdot v_i = F_i \cdot v_i = F_i \cdot \frac{dr_i}{dt}.
\]

Mathematicians hate when we do this, but let’s write this relationship as

\[
dK_i = F_i \cdot dr_i.
\]

Over time, consider that particle \( i \) moves from position \( a \) to position \( b \) along a given path. If we integrate along the path,

\[
K_i(b) - K_i(a) = \int_a^b F_i \cdot dr_i = W_{ab}
\]

where we define the integral as \( W_{ab} \), the total **work** done by the force on the particle over the path from \( a \) to \( b \).

The above is called the **work-energy theorem**, and it’s essentially Newton’s 2nd law, but expressed in “energy language.” A force exerted on a particle gives it an acceleration \( \implies \) work done on a particle gives it kinetic energy.

Work is a scalar, but it has a sign: \( W > 0 \) means net work was done ON the particle, so the kinetic energy at the *end* of the path is higher than at the beginning. If \( W < 0 \), it means work was done BY the particle ON the environment, so its kinetic energy must decrease.

(Notice that work is done only when there’s a nonzero component of the force projected *parallel to* the particle’s path. If \( \mathbf{F} \) is perpendicular to the path, you can change a particle’s direction, but no work is done... so you don’t change the particle’s kinetic energy!)
By itself, the work-energy theorem is NOT conservation of energy. A system’s total $K_i$ can change over time!

The above was true for any kind of force. However, what comes next is true for only some forces...

Gravity is a “conservative” force, which means that $W_{ab}$ is the same, no matter what path is chosen between $a$ and $b$.

(One can think about non-conservative forces, like friction...)

In this case, we can take point $a$ to be any arbitrary $r$, and take point $b \to \infty$ (i.e., very far away from any neighboring bodies), so that the force of gravity on particle $i$ there is zero.

Then, we define the particle’s potential energy function such that

$$U_i(r) = W_{r\infty} = \int_r^\infty \mathbf{F}_i \cdot d\mathbf{r}_i'$$

i.e., the potential energy at $r$ = the work done by a conservative force to bring the particle from $r$ to $\infty$.

Because conservative forces are path-independent, then

$$(W_{ab} + W_{b\infty}) \text{ ought to always be equal to } W_{a\infty}.$$  

Thus,

$$W_{ab} = W_{a\infty} - W_{b\infty} = U_i(a) - U_i(b)$$

and, for the above finite path ($a \to b$), the work done by a conservative force is equal to minus the change in potential energy associated with that force.

So, if

$$K_i(b) - K_i(a) = U_i(a) - U_i(b) \text{ , then } K_i(a) + U_i(a) = K_i(b) + U_i(b)$$

i.e., conservation of total (kinetic + potential) energy for the particle.
What is the total energy of our two-body system? We can see that

\[ E = \sum_{i=1}^{2} (K_i + U_i) = \text{constant over time}. \]

Each particle has its own kinetic energy \( K_i \), so their total is just a simple sum. When we work out the sum over the \( U_i \) terms, something interesting happens:

\[ U_{\text{tot}} = \int_{r}^{\infty} (\mathbf{F}_{12} \cdot d\mathbf{r}_1 + \mathbf{F}_{21} \cdot d\mathbf{r}_2) \]
\[ = \int_{r}^{\infty} \mathbf{F}_{12} \cdot (d\mathbf{r}_1 - d\mathbf{r}_2) = \int_{r}^{\infty} \mathbf{F}_{12} \cdot d(\mathbf{r}_1 - \mathbf{r}_2). \]

If we use the shorthand \( \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \), then

\[ \mathbf{F}_{12} = -\frac{G m_1 m_2}{|\mathbf{r}|^3} \quad \text{so} \quad U_{\text{tot}} = -G m_1 m_2 \int_{r}^{\infty} \frac{\mathbf{r} \cdot d\mathbf{r}}{|\mathbf{r}|^3} \]

and if we’re correct that the choice of path doesn’t matter, then let’s just assume \( \mathbf{r} \) and \( d\mathbf{r} \) are parallel, and we integrate outwards along the line-of-centers between the two particles:

\[ U_{\text{tot}} = -G m_1 m_2 \int_{r}^{\infty} \frac{dr'}{(r')^2} = \frac{G m_1 m_2}{\infty} - \frac{G m_1 m_2}{r} = -\frac{G m_1 m_2}{r} \]

and the two-body system has only one “mutual” potential energy term.

(An \( N \)-body system will have \( N(N-1)/2 \) potential energy terms, corresponding to the number of unique pairings between the bodies.)

Thus,

\[ E = \frac{1}{2} m_1 |v_1|^2 + \frac{1}{2} m_2 |v_2|^2 - \frac{G m_1 m_2}{r} = \text{constant over time}. \]
Center-of-Mass Coordinates

We’ve learned some useful things from the conservation laws, and their application will become even simpler if we implement a change of variables.

First, let’s write down the full set of dynamics equations that we’ll want to solve (taking into account the equal-and-opposite forces):

\[
\begin{align*}
\frac{dr_1}{dt} &= v_1 \\
\frac{dr_2}{dt} &= v_2 \\
\frac{dv_1}{dt} &= -\frac{G m_1 m_2 (r_1 - r_2)}{|r_1 - r_2|^3} \\
\frac{dv_2}{dt} &= -\frac{G m_1 m_2 (r_2 - r_1)}{|r_2 - r_1|^3}.
\end{align*}
\]

Now let’s define the center-of-mass (CM) position and velocity vectors,

\[
R = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} \quad V = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}
\]

and let’s formalize the relative position and velocity vectors,

\[
r = r_1 - r_2 \quad v = v_1 - v_2.
\]

If we can solve for \( R \) and \( r \), we can always use algebra to solve for the original coordinates:

\[
r_1 = R + \frac{m_2 r}{m_1 + m_2} \quad r_2 = R - \frac{m_1 r}{m_1 + m_2}
\]

and similarly for the velocities.

If we add together the two \( F = ma \) equations of motion, we find

\[
\frac{dV}{dt} = 0 \quad \text{i.e.,} \quad V = \text{constant over time.}
\]

i.e., no matter what happens between the two bodies (orbit, flyby, collision), the CM just keeps chugging along at a constant speed.

(This was essentially the same calculation as proving \( dp_{\text{tot}}/dt = 0 \).)
This means that if we “transform” into the CM reference frame, then the only variables that vary in time are the relative ones (r and v).

Let’s try combining the two $F = ma$ equations in a different way.

Take $m_1 \times \{ \text{the } v_2 \text{ equation} \}$, and subtract it from $m_2 \times \{ \text{the } v_1 \text{ equation} \}$, and we get...

$$\mu \frac{dv}{dt} = \frac{-Gm_1 m_2 r}{|r|^3}$$

where the reduced mass is $\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$.

It’s often easier to remember it as $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$.

Reduced mass measures the “inertia” of what’s in motion in the CM frame.

Note that if particle $m_1$ is so massive that it doesn’t move (i.e., $m_1 \gg m_2$), then $\mu \approx m_2$.

If the two particles have the same mass ($m_1 = m_2 \equiv m$), then $\mu = m/2$.

Notice something else: A few steps of algebra will show that

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \sim \sim = \frac{1}{2} MV^2 + \frac{1}{2} \mu v^2$$

where we define the total mass as $M = m_1 + m_2$. Thus, the total kinetic energy has two terms: one for the motion of the CM itself, and one for relative motions within the CM.

Since we showed above that the first term is constant and unchanging, we usually leave it out of the total energy $E$.

$$E = \frac{1}{2} \mu v^2 - \frac{Gm_1 m_2}{r} = \text{constant over time}.$$  

After all, the main reason we use $E$ is to “constrain” how its component parts change over time. If there’s a piece of it that never changes, who needs it?

ALSO, one can examine the total angular momentum and show that, if we transform into the CM reference frame,

$$L_{\text{tot}} = r_1 \times m_1 v_1 + r_2 \times m_2 v_2 = \sim \sim = \mu r \times v.$$
However, if we remember that \( \mathbf{L}_{\text{tot}} \) is constant in time (i.e., always pointing in the same fixed direction), then this means \( \mathbf{r} \) and \( \mathbf{v} \) must always stay in the \textbf{same 2D plane} perpendicular to \( \mathbf{L}_{\text{tot}} \).

For convenience, let’s assign the \( z \)-axis to the direction of \( \mathbf{L}_{\text{tot}} \), so that \( \mathbf{r} \) and \( \mathbf{v} \) remain in the \( xy \)-plane. Let’s also use cylindrical coordinates, which are just polar coordinates in the \( xy \)-plane, with

\[
\begin{align*}
\mathbf{r} &= r \hat{e}_r \\
\mathbf{v} &= (\dot{r}) \hat{e}_r + (r \dot{\phi}) \hat{e}_\phi
\end{align*}
\]

and we’ll call the \( z \)-component of the total angular momentum:

\[
\ell = \mu r^2 \dot{\phi} = \text{constant}.
\]

And, to sum up, we also have:

\[
E = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\phi}^2) - \frac{Gm_1m_2}{r} = \frac{1}{2} \mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} - \frac{Gm_1m_2}{r} = \text{constant}.
\]

The original two-body problem has essentially been reduced to an equivalent \textbf{one-body problem}, which involves a single “object” with mass \( \mu \) a distance \( \mathbf{r} \) away from an origin at the CM. (Despite the fact there’s nothing at the origin, this fictitious object feels a central force towards it.)

\[
\mathbf{v} = \frac{d\mathbf{r}}{dt} \quad \& \quad \mu \frac{d\mathbf{v}}{dt} = -\frac{Gm_1m_2 \mathbf{r}}{|\mathbf{r}|^3}
\]

Even though we haven’t solved any equations of motion yet, we now know enough to derive \textbf{Kepler’s 2nd law} (i.e., equal areas in equal times)!

Strangely, it’s more fundamental than Kepler’s 1st law, in that it \textit{doesn’t} depend on the precise functional form of the Newtonian potential energy. (It’s still true, say, in Einstein’s general relativity!)
Let’s define the area $A$ swept out by the position vector $\mathbf{r}$ of the particle’s path (in the CM frame) between time $t$ and $t + dt$:

For very short times, $\mathbf{r}(t) \approx \mathbf{r}(t + dt)$, so the triangle has area

$$dA = \frac{1}{2} r(r \, d\varphi) = \frac{1}{2} r^2 d\varphi .$$

(assuming $d\varphi \ll 1$)

Thus,

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\varphi}{dt} = \frac{1}{2} r^2 \dot{\varphi} = \frac{\ell}{2\mu} = \text{constant} .$$

Thus, Kepler’s 2nd law (planets sweep out “constant areas over constant times”) is really just a consequence of angular momentum conservation. $\ell \propto r^2 \omega$, so as $r \downarrow$, orbits must speed up.

Keplerian Orbits

Our goal is a complete solution for $\mathbf{r}(t)$, i.e., $r(t)$ and $\varphi(t)$.

There are several ways to proceed. First, let’s just look at the consequences of energy conservation.

Solve $E = \text{constant}$ for $\dot{r}$ and we get a differential equation:

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{\mu} \left[ E + \frac{Gm_1m_2}{r} - \frac{\ell^2}{2\mu r^2} \right]} = \sqrt{\frac{2}{\mu} \left[ E - V(r) \right]}$$

where one often sees the effective potential

$$V(r) = -\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}$$

as the sum of the gravitational potential a centrifugal potential (corresponding to what some call a “fictitious force”). We’ll plot this function soon.
Assuming we know the constants $E$ & $\ell$, we could:

- Solve for $dt$ & integrate to get $t(r)$
  
  $\frac{dr}{dt} = f(r) \implies \int \frac{dr}{f(r)} = \int dt = t$.

- Invert the solution to get $r(t)$.

- Integrate the definition of $\ell$ to get $\varphi(r) \rightarrow \varphi(t)$.

In general, this whole process cannot be done analytically. However, from the 1700s to the 1900s, a huge amount of effort was spent to find closed-form solutions. I won’t go into Kepler’s equation (which depends on $\varphi$-like quantities called the “mean anomaly” & “eccentric anomaly”) and is probably still taught in celestial navigation classes.

Thus, let’s look again at the equation of motion. The only vector component with non-zero terms is along the line of centers between the bodies; i.e., $\hat{e}_r$:

$$\mu a_r = \mu(\ddot{r} - r\dot{\varphi}^2) = -\frac{Gm_1m_2}{r^2}.$$  

(Note: for $N = 3$ bodies or more, transverse accelerations appear! This is what makes those problems way more complicated than $N = 2$.)

We can rearrange this a bit by using the definitions $\ell = \mu r^2 \dot{\varphi}$ and $\mu = m_1m_2/(m_1 + m_2)$ and $M = m_1 + m_2$, 

$$\ddot{r} = \frac{\ell^2}{\mu^2 r^3} - \frac{GM}{r^2}.$$  

In the 1800s, Jacques Phillipe Binet found a clever change of variables to implement at this point: $u = 1/r$. Using that, one can write

$$\frac{du}{d\varphi} = \frac{du}{dr} \frac{dr}{d\varphi} = -\frac{1}{r^2} \frac{dr}{dr} = -\frac{\dot{r}}{r^2 \dot{\varphi}} = -\frac{\mu \dot{r}}{\ell}.$$  

Taking a second derivative, and using the same tricks, we get

$$\frac{d^2u}{d\varphi^2} = -\frac{\mu^2 r^2 \ddot{r}}{\ell^2}.$$  

2.12
Solve this for \( \ddot{r} \), and plug into the equation of motion, and we now get
\[
\frac{d^2u}{d\varphi^2} = -u + \frac{GM\mu^2}{\ell^2} \quad ("\text{Binet's equation"})
\]

Why did we do all this? Notice that we’ve eliminated time in favor of an ODE that when solved will give \( u(\varphi) \)... and that gives \( r(\varphi) \): the shape of the orbit in space.

Also, note that the final term in Binet’s equation is a constant. We can perform yet another change of variables:
\[
y = u - \frac{GM\mu^2}{\ell^2} \implies \text{Binet’s equation becomes } \frac{d^2y}{d\varphi^2} + y = 0
\]

Have you seen this 2nd order ODE before? The solution is a sinusoid (i.e., pretty much any linear combination of \( \sin \varphi \) and \( \cos \varphi \) you can think of).

A general version of the solution can be written as
\[
y(\varphi) = y_0 \cos(\varphi - \varphi_0)
\]
which needs us to specify two constants of integration.

Converting back to “real” units \( (y \to u \to r) \),
\[
r(\varphi) = \frac{\ell^2/(GM\mu^2)}{1 + (\ell^2 y_0)\cos(\varphi - \varphi_0)/(GM\mu^2)} = \frac{r_c}{1 + e \cos(\varphi - \varphi_0)}
\]
where the new constants are
\[
r_c = \frac{\ell^2}{GM\mu^2} \quad (\text{radius of curvature}) \quad e = \frac{\ell^2 y_0}{GM\mu^2} \quad (\text{eccentricity})
\]

*Note:* \( r_c \) tells us the overall spatial scale of the orbit, while \( e \) tells us more about its shape. \( \varphi_0 \) sets the overall orientation of the orbit.

This solution describes conic sections: circles, ellipses, parabolas, and hyperbolas.

Let’s explore what these solutions look like in the context of total energy conservation. We can take the solution \( r(\varphi) \) and substitute it into
\[
E = \frac{1}{2}\mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} - \frac{Gm_1m_2}{r} \quad \left(\text{using } \dot{r} = \frac{dr}{d\varphi} \dot{\varphi} \right).
\]
Thus, we’re able to solve for $e$ as a function of total energy:

$$e = \sqrt{1 + \frac{2r_cE}{Gm_1m_2}} \quad \text{(a nicer way to write } e\text{)}.$$

What do the orbits look like... and how does $E$ compare to the effective potential $V(r)$?

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<td>$&gt;0$</td>
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<td>$(v &gt; 0$ as $r \to \infty$)</td>
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<td>$=1$</td>
<td>$=0$</td>
<td>parabola</td>
<td>$(v = 0$ as $r \to \infty$)</td>
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<td>$=0$</td>
<td>$=V_{\text{min}}$</td>
<td>circle</td>
<td>($r = r_c$)</td>
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<tr>
<td>$&lt;0$</td>
<td>$&lt;V_{\text{min}}$</td>
<td>not allowed</td>
<td>($v^2 &lt; 0$, imaginary velocity!)</td>
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where some algebra can be used to show that

$$r_{\text{min}} = \frac{r_c}{1 + e} \quad \quad r_{\text{max}} = \frac{r_c}{1 - e} \quad \quad V_{\text{min}} = -\frac{Gm_1m_2}{2r_c}.$$

Note that the plot for $V(r)$ is only for a single value of $\ell$. There’s really a whole family of $V(r, \ell)$ for all possible orbits between 2 bodies of known masses.
Kepler himself thought a lot about the **elliptical case** (1st law).

\[ a = \frac{Gm_1m_2}{2|E|} = \text{semi-major axis} \]

\[ b = \frac{\ell}{\sqrt{2\mu|E|}} = \text{semi-minor axis} \]

\[ \text{Area} = \pi ab, \quad \frac{b}{a} = \sqrt{1-e^2}, \quad r_c = \frac{b^2}{a} \]

The orbit around one focus ranges between the **apsides**:

\[
\begin{align*}
\{ & \text{periapsis / pericenter / perigee} \} \\
& \{ \text{apapsis / apocenter / apogee} \} \\
\end{align*}
\]

\[ r_{\text{min}} = a(1-e) = r_c/(1+e) \]

\[ r_{\text{max}} = a(1+e) = r_c/(1-e) \]

We can also derive **Kepler’s 3rd law** by recalling the 2nd law:

\[ dt = \frac{2\mu}{\ell} dA \quad \text{(for an ellipse).} \]

Both sides can be integrated over an exact period:

\[
\left\{ \begin{array}{l}
\text{At } t = 0 \rightarrow P \\
\text{At } A = 0 \rightarrow \pi ab \\
\end{array} \right.
\]

Thus,

\[ P = \frac{2\mu}{\ell} \pi ab = \frac{2\mu}{\ell} \pi a^{3/2} \sqrt{r_c} = \frac{2\mu}{\ell} \pi a^{3/2} \sqrt{\frac{\ell^2}{GM\mu^2}}. \]

Kepler squared both sides. The \( \ell \)'s and \( \mu \)'s cancel, and

\[ P^2 = \left( \frac{4\pi^2}{GM} \right) a^3 \]

which is essentially what Kepler had in mind, except that he didn’t realize that \( M \) isn’t a universal constant for every planet/Sun pair. However, in our solar system,

\[ M = m_1 + m_2 = M_{\text{sun}} + M_{\text{planet}} \approx M_{\text{sun}} \]

so the 3rd law is **very close** to being exact.

For circular orbits, \( r = a \), and astronomers often express it as

\[ \omega = \frac{2\pi}{P} = \sqrt{\frac{GM}{r^3}} \propto r^{-3/2} \quad \text{(the “Keplerian orbital frequency”).} \]
We’ll soon examine a few interesting applications of the two-body problem. But first, there’s one more mathematical tool to derive.

The total energy conservation equation can be solved for velocity:

\[ E = \frac{1}{2} \mu v^2 - \frac{G m_1 m_2}{r} \quad \implies \quad v^2 = \frac{2E}{\mu} + \frac{2GM}{r}. \]

Using the expressions derived for ellipses, this reduces to

\[ v^2 = GM \left( \frac{2}{r} - \frac{1}{a} \right) \]

(the “vis-viva” equation).

Applications:

1. Measuring exoplanet & black-hole masses from radial velocities (ellipses)
2. Rocket-flight: efficient orbit transfer with minimum fuel (ellipses)
3. How big are protostellar accretion disks? (parabolae)
4. Rocket flight: the gravitational slingshot effect (hyperbolae)
5. How does our solar system orbit the center of the galaxy? (N/A?!)
(1) Binary Radial Velocities

When observing circular/elliptical orbits from afar, we’re essentially back in the inertial frame, so we’re dealing with the individual velocities for bodies 1 and 2:

\[ v_1 = V + \frac{m_2 v}{m_1 + m_2} \quad v_2 = V - \frac{m_1 v}{m_1 + m_2} \]

The two bodies oscillate around the CM, 180° out of phase with one another.

Here is a gallery of elliptical orbits plotted by Adrian Price-Whelan (@adrianprw), varying both e & mass ratio \( q = m_2/m_1 \), converted back to the inertial frame:
For velocities observed via Doppler shift (which we’ll talk more about later), what we “see” is just a projected line-of-sight component of the vector.

In the simple case of circular orbits \((e = 0)\), that component varies sinusoidally in time...

(Values of \(v_1 \& v_2\) in equations below are the \textbf{maximum} values along these sinusoid curves.)

If we see signatures of BOTH components in the spectrum, it’s straightforward to figure out their masses \(m_1\) and \(m_2\). First, the CM coordinate conversion formula above indicates

\[
\frac{m_1}{m_2} = \frac{v_2}{v_1}
\]

and even if we only measure \(v_{1,LOS}\) and \(v_{2,LOS}\), the \(\sin i\)’s cancel out of this ratio.

Also, Kepler’s 3rd law gives us the total mass:

\[
m_1 + m_2 = \frac{4\pi^2}{GP^2} a^3.
\]

Can we measure \(a\)? It’s easy if it’s an astrometric binary (e.g., with positions trackable on the sky). Even if it’s not, the assumption of a circular orbit helps:

\[
v_1 = \frac{2\pi r_1}{P} \quad \quad v_2 = \frac{2\pi r_2}{P}
\]

so we can solve for

\[
a = r_1 + r_2 = \frac{P}{2\pi}(v_1 + v_2) \quad \text{and thus} \quad m_1 + m_2 = \frac{P}{2\pi G}(v_1 + v_2)^3.
\]

If we know \(i\), everything on the right is observable.
If we don’t know $i$, then what we’ve really got is
\[
m_1 + m_2 = \frac{P}{2\pi G} \frac{(v_{1,\text{LOS}} + v_{2,\text{LOS}})^3}{\sin^3 i}
\]
and we can try a range of guesses for $i$.

If we see only signatures of ONE component in the spectrum (say, $v_1$) then we’re not out of luck. Replace $v_2$ by $(m_1 v_1/m_2)$, and Kepler’s 3rd law is
\[
m_1 + m_2 = \frac{P}{2\pi G} \frac{v_{1,\text{LOS}}^3}{\sin^3 i} \left(1 + \frac{m_1}{m_2}\right)^3
\]
and this can be rearranged to solve for the binary “mass function:”
\[
\frac{m_2^3 \sin^3 i}{(m_1 + m_2)^2} = \frac{P}{2\pi G} \frac{v_{1,\text{LOS}}^3}{\sin^3 i} \equiv f
\]
which is measurable. Let’s examine two limiting cases:

- If the unseen thing is an exoplanet ($m_2 \ll m_1$), then
  \[
m_2 \approx \frac{m_1^{2/3} f^{1/3}}{\sin i} \quad m_2 > m_1^{2/3} f^{1/3}.
\]

- If the unseen thing is a supermassive black hole ($m_1 \ll m_2$), then
  \[
m_2 \approx \frac{f}{\sin^3 i} \quad m_2 > f.
\]

Dozens of stars near our own galaxy’s SMBH have been tracked for decades (both astrometrically & spectroscopically), and their orbits tell us the central black hole must have a mass of $\sim 4 \times 10^6 M_\odot$.

(Star S0-102 orbits with a period of only 11.5 years, even though its $a \approx 400$ AU...)

In both cases, we can obtain a firm lower limit on $m_2$ (though for the exoplanet case it also requires an estimate of $m_1$).
For eccentric orbits, the observable radial velocity curves are no longer sinusoids, and their shapes also depend on precisely how the system is oriented with respect to the LOS...

![Diagram](image)

(2) Rocket Orbit Transfer

If you’re in one circular or elliptical orbit, and you’d like to get to a different one, there are multiple ways you can fire your rockets to do it.

The most efficient way is the **Hohmann transfer orbit**, which lets you do it with the minimum thrust.

What do we mean by “thrust?” A given rocket burns with essentially a known & constant force $F$. If you burn it for a time $\Delta t$, then turn it off, you’ve changed your velocity by a given amount:

$$F = \frac{\Delta p}{\Delta t} = \frac{m_R \Delta v}{\Delta t} \implies \Delta v = \frac{F \Delta t}{m_R}$$

where $m_R$ is the current mass of the rocket. To accelerate, you point the rocket behind your current velocity vector $v$. To decelerate, you point the rocket along $v$.

So how do you choose the $\Delta v$ that will get you to your new orbit? **Vis-viva!**

Let’s say we’re in a “low” circular orbit around the Earth with radius $r_1$. We want to get to a higher circular orbit with $r_2 > r_1$. 

2.20
There must be two rocket burns:

1. Boost from the low circular orbit to an elliptical orbit with the same perigee ($r_1$), and an apogee of $r_2$.
2. Once you reach the apogee of $r_2$, boost again to change the orbit from elliptical to circular.

To determine the required $\Delta v$ for each step, just solve the vis–viva equation for the speeds...

The first burn:

$$\Delta v_1 = v(\text{ellip. at } r_1) - v(\text{circ. at } r_1)$$

and it’s straightforward to use vis-viva to evaluate

$$v(\text{circ. at } r_1) = \sqrt{\frac{GM}{r_1}} .$$

The other term requires us to recognize that $2a = r_1 + r_2$, so that

$$v(\text{ellip. at } r_1) = \sqrt{\frac{GM}{\frac{2}{r_1} - \frac{2}{r_1 + r_2}}} = \sqrt{\frac{2GMr_2}{r_1(r_1 + r_2)}}$$

and thus

$$\Delta v_1 = v(\text{ellip. at } r_1) - v(\text{circ. at } r_1) = \sqrt{\frac{GM}{r_1}} \left[ \sqrt{\frac{2r_2}{r_1 + r_2}} - 1 \right] .$$

Similar math could be done for the second burn, but we won’t go through it.

It’s time to point out something strange about Keplerian orbits. Notice that we had to speed up (twice!) to get from the $r_1$ circular orbit to the $r_2$ circular orbit. However, the orbital speed is slower at $r_2$! Recall the orbital frequency:

$$\omega = \sqrt{\frac{GM}{r^3}} \quad \Rightarrow \quad v_{\text{circ}} = \omega r = \sqrt{\frac{GM}{r}}$$

and as $r$ goes up, $v_{\text{circ}}$ goes down.
However, a circular orbit with larger $r$ has a larger angular momentum

$$\ell = \mu r^2 \dot{\phi} = \mu r^2 \omega = \mu r^2 \sqrt{\frac{GM}{r^3}} = \mu r^2 \sqrt{\frac{GM}{r}}$$

and a larger (i.e., less negative) total energy

$$E = V_{\text{min}} = -\frac{Gm_1m_2}{2r_c}$$

(and for a circular orbit, $r = r_c$)

so as $r$ increases, both $\ell$ and $E$ increase, too.

It’s counter-intuitive, but it’s how the physics works. If you’re in orbit around the Earth at a given radial distance, and you wanted to “pass” a satellite in a neighboring orbit (i.e., blow by it at a higher speed), you’d have to:

- Fire your rockets in a forward direction, to slow down.
- This decreases your $\ell$ and $E$ and drop you into a lower orbit,
- in which you’ll have a faster orbital speed!

Maybe a plot will be helpful:

By going to a lower radius, you end up speeding up (higher $K$), but you’ve lost total energy (i.e., $E$ is lower = more negative). The only way to do that is to do “negative work.”

\[ K = \frac{1}{2} \mu v^2 = +\frac{Gm_1m_2}{2r} \]

\[ E_{\text{total}} = -\frac{Gm_1m_2}{2r} \]

\[ U_G = -\frac{Gm_1m_2}{r} \]
Gravity inevitably pulls stuff together. Consider a Giant Molecular Cloud (GMC) in the disk of our galaxy. They become turbulent and break into protostellar “cores” that will eventually collapse into stars. Typical properties:

\[
R_{\text{core}} \approx 0.1 \text{ parsec} \\
M \approx 1 \text{ solar mass} \\
\omega \approx 10^{-15} \text{ rad/s}
\]

Rotation is slow; it only gets a gentle kick from galactic shear motions:

\[
\mathcal{P} = \frac{2\pi}{\omega} = 200 \text{ million years. } \quad v_{\text{circ}} = \omega R_{\text{core}} = 0.003 \text{ km/s.}
\]

Let’s assume there’s already a protostar forming at the center of the GMC, which dominates the total mass \( M \), and has a radius \( R_\odot \ll R_{\text{core}} \).

Because of the slow rotation of the cloud, it’s a good approximation to assume that a small clump of gas from its outer edge falls inwards on a parabolic orbit; i.e., with \( E \approx 0 \), and thus zero kinetic energy at its outer starting point (essentially \( r \to \infty \)).

We want to know: **Will the clump impact the star?**

The answer depends on how much angular momentum it has. Assume \( m_1 \approx M \) is the star and \( m_2 \ll M \) is the clump, then \( \mu \approx m_2 \). We can set the angular momentum at the outermost “initial condition:”

\[
\ell = \mu r^2 \dot{\varphi} = \mu R_{\text{core}}^2 \omega = \text{constant.}
\]

For a parabolic orbit with \( e = 1 \),

\[
r(\varphi) = \frac{r_c}{1 + \cos \varphi}
\]

and the radial distance of closest-approach to the star will occur at \( \varphi = 0 \), or

\[
r_{\text{min}} = \frac{r_c}{2} = \frac{\ell^2}{2GM \mu^2} = \frac{\omega^2 R_{\text{core}}^4}{2GM}
\]

Plugging in the above numbers: \( r_{\text{min}} \approx 490 R_\odot \approx 2.3 \text{ AU} \).

Thus, **NO**, most parcels won’t impact the star, because they’ve got too much angular momentum and are unable to move so far in.
Instead, many infalling parcels end up interacting with one another, usually when they try to pass through the rotational midplane:

There are really many parcels, all coming in with random values of $\alpha$ between 0 and $2\pi$.

Infall may start as random, but if there’s a net overall sense of non-radial (rotational) flow, the north/south motions can cancel out, and the east/west motions remain to flow in one predominant direction.

The flow flattens into an **accretion disk**, and our derived size of “a few AU” is close to what is observed.

Interestingly, vis-viva says the parcels first cross the disk-plane (at $\varphi = \pm \pi/2$) with $v^2 = 2GM/r_c$, but they become rapidly “circularized” into a Keplerian orbit at that distance, which has $v^2 = GM/r_c$.

Roughly $\sim$half of their kinetic energy went into **heating up** the gas in the disk!

This is a **frictional/viscous** effect that ultimately causes the orbiting parcels to lose energy, and thus spiral into the star very slowly. (*That’s why it’s called an accretion disk, and not an orbiting-forever disk...*)
(4) The Gravitational Slingshot Effect

You’re in a rocket, and you want to change your speed by some $\Delta v$. However, you don’t have enough fuel. What do you do...?

There’s hope. Consider a hyperbolic flyby between a spacecraft and a planet or moon. The eccentricity determines the “opening angle” of the hyperbola:

These trajectories show the relative motions described by $\mathbf{r}$ & $\mathbf{v}$.

There’s one interesting fact to learn about $\mathbf{v}$ in hyperbolic orbits. Consider the “initial” and “final” conditions (way before & way after the closest approach).

For both conditions, $r \to \infty$, so the potential energy $U_{\text{tot}}$ is essentially zero in both places. Thus,

$$E = \frac{1}{2} \mu |\mathbf{v}|^2 - \frac{Gm_1m_2}{r} \approx \frac{1}{2} \mu |\mathbf{v}|^2 = \text{constant}$$

so we see that $|\mathbf{v}_{\text{init}}| = |\mathbf{v}_{\text{final}}|$ in the CM frame.

However, we’re going to have to transform back into the inertial frame. Let’s assign $m_1 =$ spacecraft, and $m_2 =$ planet. Thus, $m_2 \gg m_1$. The velocity of the CM frame is

$$\mathbf{V} = \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2} \approx \mathbf{v}_2$$

i.e., the planet dominates the total momentum of the system.

Rather than just jump into the CM frame, we should be aware that the planet is in orbit around the Sun. Thus, over short time intervals, we can consider $\mathbf{v}_2 \approx \mathbf{V}$ to be a known planetary orbital velocity.
Anyway, we see that

\[ \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 \approx \mathbf{v}_1 - \mathbf{V} \]

i.e., the relative velocity \( \approx \) velocity of spacecraft in CM frame.

Let’s assume we’re in a low-eccentricity hyperbolic orbit, and let’s remember that \( |\mathbf{v}_{\text{init}}| = |\mathbf{v}_{\text{final}}| \).

\[ \mathbf{v}_{1,\text{init}} \equiv -S \hat{e}_x \quad (\text{let’s set } S > 0) \]

\[ \mathbf{v}_{\text{init}} = \mathbf{v}_{1,\text{init}} - \mathbf{V} = (-S - V) \hat{e}_x \]

\[ \mathbf{v}_{\text{final}} = -\mathbf{v}_{\text{init}} = (S + V) \hat{e}_x \]

Thus, going back to the inertial frame,

\[ \mathbf{v}_{1,\text{final}} = \mathbf{v}_{\text{final}} + \mathbf{V} = (S + 2V) \hat{e}_x \]

i.e., the \( x \) magnitude of the spacecraft velocity changed from \( S \) to \( S + 2V \).

If the spacecraft approaches the planet “head-on” in the planet’s orbit (i.e., \( V > 0 \)), the spacecraft speeds up after the slingshot.

- Similar to the terrestrial analogy of a tennis ball being thrown at an approaching wall... it bounces back faster (see also “Fermi acceleration”).

If the spacecraft approaches the planet “along” the planet’s orbit (i.e., \( V < 0 \)), the slingshot slows down the spacecraft.

- This is how \textit{Parker Solar Probe} is inching its way closer to the Sun... by using Venus to shed its angular momentum.
(5) Orbits in a Galactic Potential

Even though there’s a lot of mass close to the center of our Milky Way Galaxy, it’s not right to think of our Sun orbiting it like a planet orbits a star.

There’s lots of mass spread out over a huge volume...

It’s beyond the scope of this course to derive the gravitational potential energy function of a galaxy, but we can work with it, if we know it!

Let’s say we know the $U(r)$ felt by low-mass “test-particles” in the equatorial plane of the galaxy. When we first derived $U$, we took it as the integral of the force over a known path.

Thus, it shouldn’t be surprising to see that one can obtain the force as the derivative of $U$:

$$F = -\frac{dU}{dr} \hat{e}_r$$

and you can verify it for yourself with the point-mass case $U = -\frac{Gm_1m_2}{r}$.

Since $F = ma$ (where $m$ is the mass of the tiny test-particle feeling the force), let’s divide both sides by $m$ to get the gravitational acceleration as a function of a potential

$$a = -\frac{d\Phi}{dr} \hat{e}_r$$

and $\Phi = U/m$ is more fundamental. It describes only the galaxy, whereas $U$ has to describe both the galaxy and the test-particle.

If we know $\Phi(r)$, we can do several things with it.

First, we can compute the circular velocity of a test particle at a given distance $r$. Recall the $r$-component of our one-body equation of motion:

$$\mu(\ddot{r} - r\dot{\varphi}^2) = F_r.$$
For circular orbits, \( \ddot{r} = 0 \), and we can write \( \omega = \dot{\phi} \). Also, take the equation of motion and divide both sides by the test-particle mass \( \approx \mu \),

\[
    r \omega^2 = a_r = \frac{d \Phi}{dr}
\]

This is really the same thing you did in Physics 1, when you balanced gravity with the “centripetal force” to learn about circular motion.

In any case, \( \omega = \sqrt{\frac{1}{r} \frac{d \Phi}{dr}} \) and \( V_{\text{circ}} = \omega r = \sqrt{r \frac{d \Phi}{dr}} \).

Second, we can compute the escape velocity of a test particle at a given distance \( r \). Energy conservation is a helpful concept here:

\[
    E = \frac{1}{2} \mu v^2 + U \quad \Rightarrow \quad \frac{E}{\mu} = \frac{v^2}{2} + \Phi = \text{constant}.
\]

Recall the effective potential plot for Keplerian orbits. To reach \( r \to \infty \), one needs at least \( E \geq 0 \).

In other words, at infinity, we know that \( \Phi \to 0 \) (because we’re infinitely far from the sources of gravity). We could launch with a huge velocity and get to infinity with a positive speed (i.e., hyperbolic), but the \textit{minimum} launch speed would get us to infinity with zero speed (i.e., parabolic).

This means we’re dealing with \( E = 0 \). If that’s the case, then at any arbitrary \( r \), we would require an effective escape velocity of

\[
    v^2 = -2\Phi \quad \Rightarrow \quad V_{\text{esc}} = \sqrt{-2\Phi}.
\]

There’s been several decades of improvement in the observations used to build an “empirical” model of \( \Phi(r) \) for our galaxy. My old friend Scott Kenyon published one in 2008 that summed together the potentials of 4 components:

- The supermassive black hole (“Sgr A*”) that behaves like a standard Newtonian point-mass: \( \Phi_{\text{BH}} \approx -GM_{\text{BH}}/r \).
- The central bulge, which is mostly composed of old low-metallicity stars.
- The thin disk of younger high-metallicity stars and spiral arms.
- The dark matter halo that extends out almost to our neighboring Andromeda Galaxy.
As opposed to Keplerian orbits, 

\[ V_{\text{circ}} = \sqrt{\frac{GM}{r}} \quad \text{and} \quad V_{\text{esc}} = \sqrt{\frac{2GM}{r}}, \]

the rotation curve of the galaxy is rather “flat” in the vicinity of the spiral arms. The local differential-rotation shear (which spins up GMCs) is difficult to see in this plot.

At the Sun’s orbit of \( r = 8 \text{ kpc} \), \( V_{\text{circ}} = 219 \text{ km/s} \), which implies a galactic “year” of \( 2\pi/\omega \approx 220 \text{ million years} \).

At the Sun’s orbit, \( V_{\text{esc}} = 638 \text{ km/s} \), but in the inner parts of the galaxy, \( V_{\text{esc}} > 1000 \text{ km/s} \). If we see stars in the halo with speeds this high, it usually implies they’ve been ejected hyperbolically from close encounters with Sgr A*!