Incorporating spectral characteristics of Pc5 waves into three-dimensional radiation belt modeling and the diffusion of relativistic electrons

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[1] The influence of ultralow frequency (ULF) waves in the Pc5 frequency range on radiation belt electrons in a compressed dipole magnetic field is examined. This is the first analysis in three dimensions utilizing model ULF wave electric and magnetic fields on the guiding center trajectories of relativistic electrons. A model is developed, describing magnetic and electric fields associated with poloidal mode Pc5 ULF waves. The frequency and L dependence of the ULF wave power are included in this model by incorporating published ground-based magnetometer data. It is demonstrated here that realistic spectral characteristics play a significant role in the rate of diffusion of relativistic electrons via drift resonance with poloidal mode ULF waves. Radial diffusion rates including bounce motion show a weak pitch angle dependence for $\alpha_{eq}/C_21 = 50/C_176$ for a power spectral density which is $L$-independent. The data-based model for greater power at higher $L$ values yields stronger diffusion at $\alpha_{eq} = 90^\circ$. The $L^6$ dependence of the diffusion coefficient which is obtained for a power spectral density which is $L$-independent is amplified by power spectral density which increases with $L$. During geomagnetic storms when ULF wave power is increased, ULF waves are a significant driver of increased fluxes of relativistic electrons inside geosynchronous orbit. Diffusion timescales obtained here, when frequency and $L$ dependence comparable to observations of ULF wave power are included, support this conclusion.


1. Introduction

[2] The radiation belts are a region of the Earth’s magnetosphere permanently populated by energetic particles. These particles consist of electrons and ions (mostly protons) with energies in the keV-MeV range. They are in orbits that encircle the Earth from ~1000 km above the surface to a distance of ~7 Earth radii ($R_E$), extending from the geomagnetic equator to ±50° in latitude [Tascione, 1994; Kivelson and Russell, 1995]. There is a region of reduced MeV electron flux, called the slot region, that separates the radiation belt electrons into an inner and an outer zone. This slot region is created by resonant interactions with whistler mode waves excited by electrons in the tens to hundreds of keV energy range [Lyons et al., 1972] which diffuse electrons in pitch angle into the loss cone. The inner radiation zone is considered highly stable because of the electron energies exceeding the value required for cyclotron resonant scattering into the loss cone by whistler waves and the long lifetime from collisional losses at MeV energies. This inner electron belt resides mainly below $2 R_E$ geocentric distance in the equatorial plane, or $L \leq 2$. The outer electron belt is highly variable but in quiet times starts around $L = 3.5$ and extends to $L = 6–7$. This study only deals with electrons.

[3] Particles enter the radiation belts through a variety of means and over a range of timescales. Both electrons and ions execute three different types of periodic motion as a result of the Earth’s magnetic field: They gyrate about the magnetic field lines on a timescale of milliseconds, they bounce between hemispheres on a timescale of seconds, and they drift around the Earth with timescales of a few to tens of minutes. The timescales depend on particle energy, equatorial magnetic field strength, and pitch angle. Associated with each of these three types of motion is an adiabatic invariant. A conserved invariant is one that changes very slowly compared with the periodicities of the particle
motion and therefore is assumed to be constant. The first adiabatic invariant conserves the particle’s magnetic moment, \( M = p_\perp^2 / 2mqB \), where \( p_\perp \) is the component of particle momentum instantaneously perpendicular to the local magnetic field. The second conserves the particle’s parallel momentum integrated over one bounce period, and the third conserves magnetic flux within a drift orbit. Trapping and energization can occur on a particle drift timescale because of the induction electric field produced by rapid magnetopause compression resulting from an interplanetary shock [Li et al., 1993b]. Response of the outer zone electron flux to solar wind variations can also occur over a timescale of hours to days and even weeks.

[1] O’Brien et al. [2001, 2003] did an extensive study of solar wind data, magnetospheric indicators, and in situ particle measurements. They determined that high-speed solar wind velocity [Paulikas and Blake, 1979] and high recovery phase ULF wave power in the 1–10 mHz range [Mathie and Mann, 2000] are closely associated with the production of relativistic electrons at geosynchronous orbit. This strongly suggests that a diffusive process is at work, since the ULF wave periods of minutes are comparable to the electron drift period in the hundreds of keV–MeV range at geosynchronous orbit [Elkington et al., 2003; Hudson et al., 2001; Green and Kivelson, 2001]. The high-speed solar wind may drive the growth of velocity-shear instability along the magnetopause. These low-frequency, long-wavelength perturbations of the magnetopause boundary can transfer energy to modes in the same frequency and wavelength range (ULF waves) within the magnetosphere [Miura, 1992].

[5] ULF wave activity and its effects on particle motion are linked via radial diffusion theory. The variations in magnetic and electric fields driving the drift of the trapped particles results in diffusion of particles across drift shells through the violation of the third adiabatic invariant [Schulz and Lanzerotti, 1974]. Enhanced radial diffusion resulting from asymmetries in the magnetic field can cause increases in the radiation belt fluxes over a period of hours [Elkington et al., 1999, 2003] instead of days as typically assumed. The time that it takes for diffusion to occur and how effective it is depends on the duration of increased Pc5 wave power [Mathie and Mann, 2001] and the level of enhanced ULF waves as discussed by O’Brien et al. [2003].

[6] ULF waves can be separated into two basic modes: toroidal and poloidal. Dungey [1963] derived hydromagnetic wave equations in a general axisymmetric field geometry (a dipole being a special case of this), which led to the quantitative description of these two modes with the assumption that the ionosphere is a perfect conductor. The toroidal mode is described as the mode in which the electric field is purely radial and the magnetic and velocity perturbations are azimuthal. Magnetic shells (L shells) decouple, and each shell oscillates azimuthally independent of all others in the limit of zero azimuthal mode number \( m = 0 \). The poloidal mode is described as the mode in which the electric field is azimuthal. The magnetic and velocity perturbations are contained in a meridional plane, and the oscillations in that plane decouple in the high \( m \) limit [Radoski, 1967; Hughes, 1994]. In reality the two modes are coupled at intermediate values of \( m \).

[7] In a dipole a particle responds resonantly to Fourier components located at harmonics (including the fundamental) of its drift frequency [Schulz and Lanzerotti, 1974], leading to the resonant condition \( \omega = m\omega_B \), where \( m \) is the azimuthal mode number and \( \omega_B \) is the drift frequency. Elkington et al. [1999, 2003] did an extensive study of the effects of toroidal and poloidal mode field line resonances in a compressed dipole on energetic electrons drifting in the equatorial plane. They describe the significance of using a compressed dipole instead of a symmetric dipole magnetic field generally used when analyzing radial diffusion. They show that in an asymmetric dipole an additional resonance besides \( \omega = m\omega_B \),

\[
\omega = (m \pm 1)\omega_B, \quad (1)
\]
can contribute significantly to radial diffusion. The \( \pm 1 \) factor in equation (1) incorporates the \( m = 1 \) day-night asymmetry in a compressed dipole. Note that for \( m \geq 2 \), there are two possible resonances in equation (1): one for \( m + 1 \) and one for \( m - 1 \). This leads to three possible resonances for an \( m \geq 2 \) poloidal or toroidal mode wave. Contributions from more than one azimuthal mode number, \( m \), lead to a range of possible resonances. However, Elkington [2000, Figure 4.10] showed that power is concentrated at low \( m \) numbers in global MHD simulations (see also Y. Fei et al., manuscript in preparation, 2005). The purpose of this paper is to apply this theory and extend the work done by Elkington et al. [2003] into three dimensions and to quantify the pitch angle and \( L \) dependence of the radial diffusion coefficient in a dipole with poloidal mode ULF waves. Equatorial plane results are compared to those obtained in a compressed dipole. In addition, a ULF wave power spectral density envelope has been fit to ground-based magnetometer measurements [Bloom and Singer, 1995; Mathie and Mann, 2001] to incorporate the observed dependence on \( L \) and frequency.

### 2. Model

[8] The three-dimensional dynamics of relativistic electrons are simulated using guiding center approximation equations from Brizard and Chan [1999] to track the bounce and drift motion of particles,

\[
v_d = \frac{p_\perp B^*}{m_0 q B^||} + \frac{e}{q B^||} \mathbf{b} \times M \nabla B + \frac{e}{B^||} \mathbf{E} \times \mathbf{b},
\quad (2)
\]

\[
\frac{dp||}{dt} = \frac{B^*}{B^||} \left( q\mathbf{E} - \frac{M}{\gamma} \nabla B \right),
\quad (3)
\]

where \( \gamma \) is the relativistic correction factor, \( m_0 \) is the rest mass, and

\[
\mathbf{B}^* = \mathbf{B} + \frac{cp||}{q} \nabla \times \mathbf{b},
\quad (4)
\]

\[
B^|| = \mathbf{b} \cdot \mathbf{B}^* = B \left( 1 + \frac{cp||}{qB} \mathbf{b} \cdot \nabla \times \mathbf{b} \right).
\quad (5)
\]

These equations differ from those obtained by Northrop [1963], who neglected the \( \nabla \times \mathbf{b} \) term in equation (4). This accuracy is needed to conserve energy in three dimensions.
The energy gained by an electron moving adiabatically is given by
\[ \frac{dW}{dt} = qE \cdot v_d + \frac{M dB}{\gamma dt}. \]
where \( v_d \) is defined in equation (2). Toroidal mode waves in a dipole magnetic field yield \( E_{dr} = 0 \), leading to \( dW/dt = 0 \) for equatorially mirroring particles drifting azimuthally since \( B_c \) has a node at the equator. In compressed dipole equatorial plane simulations [Elkington et al., 2003], toroidal modes produce substantially less energization than poloidal modes for the same parameters, again, because \( E_{dr} \) is reduced relative to \( E_{rd} \) by the geometry of the drift path. The second term in equation (6) has been neglected in previous simulations of ULF wave effects on radiation belt electrons [Elkington et al., 1999, 2003] while it has been included by Li et al. [1993b] in a simple nondiffusive shock-drift acceleration model on the drift timescale. This term is energy-dependent and is negligible for plasma sheet electron source population energies (tens to hundreds of keV).

For relativistic electrons with energies in the MeV range and ULF waves with magnetic perturbations of tens of nanoteslas and electric perturbations of a few mV m\(^{-1}\) the second term in equation (6) is of the same magnitude as the first term at \( L = 6.6 \), and therefore the magnetic field perturbations are retained in our model of poloidal mode effects on equatorially and nonequatorially mirroring electrons. The goal is to create a general model which can also be applied to magnetic perturbations with rise times which are fast compared to the drift period, i.e., shock-drift acceleration [Li et al., 1993b; Elkington et al., 2002] as well as effects which can be described by a radial diffusion equation investigated in this paper,
\[ \frac{\partial f}{\partial t} = L^2 \frac{\partial}{\partial L} \left[ D_{LL} \frac{\partial f}{\partial L} \right], \]
where \( \partial \theta / \partial t \ll \omega_d \). The phase space density, \( f \), is a function of energy, pitch angle, and \( L \) or, equivalently, the three adiabatic invariants [Schulz and Lanzerotti, 1974]. \( D_{LL} \) then describes the rate of radial diffusion.

The analytic fields used as input to equations (2) and (3) are based on a compressed dipole magnetosphere with model poloidal mode ULF waves,
\[ B = B_{\text{dip}} + B_c + B_{\text{pol}}, \]
\[ B_{\text{pol}} = B_{\text{r}} \hat{r} + B_{\theta} \hat{\theta}, \]
\[ E = E_{\text{pol}} \hat{\phi}. \]
In equation (8), \( B_{\text{dip}} \) is the dipole magnetic field, \( B_c \) is the compression of the magnetic field, and \( B_{\text{pol}} \) is the poloidal magnetic field, defined in equation (9) in spherical coordinates containing radial and compressional components. \( E_{\text{pol}} \) is the poloidal mode electric field (equation (10)) which is azimuthal. A zeroth-order compression of the magnetic field is assumed to model day-night asymmetry using
\[ B_c = \left( b_c \cos \phi \right) \hat{z}. \]
Azimuthal angle, \( \phi \), is taken to be zero at local noon and increases in a counterclockwise sense, and \( b_c \) is selected on the basis of measured magnetic field values. A particle drifting in such a field will drift along contours of constant magnetic field strength, characterized by
\[ L = \frac{2 \pi R^2_{\text{eq}} B_0}{\Phi}, \]
where \( L \) is the Roederer \( L \) parameter [Roederer, 1970]. \( \Phi \) is the magnetic flux enclosed by the drift shell,
\[ \Phi = \int \int \mathbf{B} \, dS, \]
calculated here by taking the first two terms in equation (8) as \( \mathbf{B} \) and the equatorial plane as \( dS \). In a dipole magnetic field, \( L \) reduces to \( r \cos^2 \lambda \) in terms of latitude, \( \lambda \), and the drift frequency reduces to
\[ \omega_d = \left( \frac{3 M e}{2 \pi R^2 L^2} \right) \left( 1 + \frac{2 M B^2}{m c^2} \right)^{-1/2} D(y) \frac{T(y)}{T(y)}, \]
where
\[ y = \sin^2(\alpha_{eq}) \] with \( \alpha_{eq} \) being the equatorial pitch angle [Schulz and Lanzerotti, 1974]. \( D(y)/T(y) \) is 1/2 for \( \alpha_{eq} = 90^\circ \) \( (y = 1) \) and decreases to 1/3 for \( y = 0 \) [see Schulz and Lanzerotti, 1974, Figure 7]. Therefore the drift frequency is maximum at the equator.

Poloidal mode ULF waves are calculated using a single vector potential component [see, e.g., Li et al., 1993a],
\[ A_\phi = A_0 F(f) H(L) J(\omega/t, \phi) \sin \theta, \]
where
\[ A_0 = \sqrt{P_0 \Delta f}, \]
\[ J(\omega/t, \phi) = \cos(m \phi - \omega t - \phi_0), \]
\[ H(L) = 10^{0.5 m u L}, \]
\[ F(f) = f^{0.5 m u}. \]
perturbations can be calculated in spherical coordinates using

$$B_{\text{pol}} = \nabla \times A_0 \dot{\Phi},$$

$$E_{\text{pol}} = -\frac{\partial}{\partial t} A_0.$$  

The resulting magnetic and electric field perturbations in spherical coordinates are

$$B_r = \frac{2A_0 F(f) H(L) J(\omega, t, \phi) \cos \theta}{r} \left[1 - 0.5m_L L \ln(10)\right],$$

$$B_\theta = \frac{-A_0 F(f) H(L) J(\omega, t, \phi) \sin \theta}{r} \left[1 + 0.5m_L L \ln(10)\right],$$

$$E_\phi = -\omega A_0 F(f) H(L) \sin \theta \sin(m_\phi - \omega t - \phi_0).$$

From Anderson [1994], compressional Pc5 waves (poloidal mode) are quasi-sinusoidal oscillations in field magnitude with periods of 5–10 min. The compressional perturbation is often accompanied by a comparable radial perturbation. Therefore both the radial and compressional components of the magnetic field are included in this model.

[13] A time-varying sinusoid (equation (17)) describes azimuthal wave propagation and depends on the azimuthal mode number, \(m\), longitude, \(\phi\), and initial phase angle, \(\phi_0\). In the present study, only an \(m = 2\) azimuthal mode number will be considered. This is consistent with previous studies [Elkington et al., 1999, 2003] and analysis of the MHD simulation fields for the 24–28 September 1998 storm period [Elkington, 2000, Figure 4.10]. The initial phase angle is calculated using a random number generator to choose a different \(\phi_0\) for each frequency. Figure 1 shows that \(E_\phi\) and \(\partial B_\theta/\partial t\) have opposite phases at a fixed point in space as required by Faraday’s law. For every time step, at each \(L\) value and latitude the frequencies are summed over the range of 0.4–10 mHz in increments of 0.1 mHz.

[14] The power spectral density (PSD) variation with \(L\) (equation (18)) is shown in Figure 2. The slope, \(m_L\), of the log of ULF wave power as a function of \(L\) is computed using results from Elkington et al. [2004, Figure 11]. They calculated the \(L\) dependence of the azimuthal electric field wave power from a three-dimensional (3-D) global MHD simulation driven by measured upstream solar wind parameters during the 24–28 September 1998 geomagnetic storm. A slope value of \(m_L = 1/3\) was obtained by fitting a line to their \(E_\phi\) radial profile, and this slope value is plotted in Figure 2. This radial-dependent power level and slope are consistent with ground-based measurements of toroidal mode magnetic field power for solar wind conditions of \(>500\) km s\(^{-1}\) [Mathie and Mann, 2001, Figure 8]. The radial trend is also consistent with satellite observations of the poloidal/compressional component measured by Active Magnetospheric Particle Tracer Explorers (AMPTE)/CCE [Anderson, 1994].

[15] The parallel mode structure is calculated using a simple cosine function as an approximation of the parallel eigenmode structure (see Figures 3 and 4). It should be noted that \(B_r\) has a node at the equator, \(\lambda = 0^\circ\), while \(E_\phi\) and \(B_\theta\) have an antinode consistent with fundamental harmonic poloidal mode waves [Hughes, 1994]. For \(m_L = 0\) the amplitudes of \(B_\theta\) and \(B_r\) are comparable, resulting in the maximum amplitude of \(|B|\) being off equator (Figure 3). For \(m_L = 1/3\) the amplitude of \(B_\theta\) is larger than the amplitude of \(B_r\). This can be attributed to the 0.5\(m_L\) \(L \ln(10)\) term in equations (22) and (23) which is larger than 1, the \(\sin\theta\) term which is close to 1 in equation (23) (equatorward of the maximum in \(B_r\) plotted in Figure 4), and the \(\cos\theta\) term which is close to zero in equation (22). Therefore the magnitude of \(|B|\) is largest at the equator and drops off as the latitude gets larger.

[16] The PSD also varies with frequency (equation (19)) where \(m_f\) is the slope of the log of ULF wave power as a function of the log of frequency. The PSD is assumed to decrease with frequency according to ground-based magnetometer observations reported by Bloom and Singer [1995],

![Figure 1](image1.png)

**Figure 1.** Time dependence of model electric and time derivative magnetic fields in the equatorial plane for \(\lambda = 0^\circ\), \(\phi = -20^\circ\), and \(m_L = 0\) at \(L = 6.6\).

![Figure 2](image2.png)

**Figure 2.** Model L shell dependence of average Pc5 power spectral density (logarithmic scale) at \(\phi = 20^\circ\), \(\lambda = 0^\circ\), and \(t = 0\). The slopes of the curves were calculated and used in equation (18). The blue curves are for \(m_L = 1/3\), and the red curves correspond to \(m_L = 0\). Note that \(m_L = 0\) electric field power yields decreasing magnetic field power with increasing \(L\) from Faraday’s law.
measurements at \(L/C0\) obtained a slope of 1.1 from ionospheric electric field measurements at \(L = 6.6\), \(t = 45\) s, and \(m_L = 0\) in a dipole.

as seen in Figure 5. The decreasing profile of PSD has a value of \(m_f = -2\) in equation (19), determined by fitting a line in Figure 5 between 0.1 and 10 mHz. This value for \(m_f\) yields a stronger frequency dependence than either Elkington et al. [2003] assumed with a flat power spectral density \((m_f = 0)\) or Holzworth and Mozer [1979], who obtained a slope of \(-1.1\) from ionospheric electric field measurements at \(L = 6\), found. At each particle position, equation (19) is evaluated at frequency intervals of 0.1 mHz from 0.4 to 10 mHz chosen to encompass the drift frequencies, \(\omega = m\omega_d\) and \(\omega = (m \pm 1)\omega_b\), and is then summed over the frequency range. The \(x\) component oscillations observed on the ground, as seen in Figure 5, correspond to the toroidal mode (east-west) oscillation in the magnetosphere which undergoes an \(\sim 90^\circ\) rotation on transmission through the ionosphere [Hughes and Southwood, 1976]. Power in the toroidal mode as well as the poloidal mode is correlated similarly with solar wind speed and rise in MeV electron flux [Mann et al., 2004], supporting the use of the toroidal mode power spectral index for the poloidal mode waves modeled here.

3. Results

[17] In each simulation, 120 electrons were evenly distributed in azimuth at a constant \(L\). To average out radial oscillations due to \(E_\theta\), 50 simulations with these same initial conditions were executed, each with a different set of \(\phi_0\) values (see equation (17)), giving 6000 particles for each set of initial conditions. The electrons were allowed to interact with global \(m = 2\) waves in a range of frequencies, 0.4–10 mHz, designed to excite all three of the resonant modes, \(\omega = m\omega_d\) and \(\omega = (m \pm 1)\omega_b\). The position and energy of each particle were recorded as a function of time as it crossed the equator during its bounce motion every 30 s. A symmetric dipole magnetic field strength of \(B_0 = 27,500\) nT was used. The simulation domain was selected to be \(x = \pm 9 R_E\), \(y = \pm 9 R_E\), and \(z = \pm 3.5 R_E\) with an inner boundary of 2 \(R_E\). Those particles drifting or bouncing beyond this range were removed from the simulation. The following criterion is checked at each time step to ensure that the particle conserved its first invariant [Chirikov, 1987]: \(\epsilon \lesssim 0.187\), where \(\epsilon\) is the maximum of the ratio of the gyroradius to the magnetic gradient scale length perpendicular or parallel to \(B\). It can also be defined as the ratio of \(\omega/\omega_b\), which is negligible for ULF wave frequencies compared to the electron gyrofrequency [Elkington et al., 2002]. It should be noted that all particles in the simulations meet this criterion at all times; therefore using a guiding center approximation is valid.

[18] Plotted in Figure 6 is the mean square deviation of \(L\), \(\langle (L - L_0)^2 \rangle\), as a function of time for electrons with an initial \(L = 6.5\) and \(M = 273\) MeV G\(^{-1}\) (corresponds to \(W = 1\) MeV at \(L = 4\)) in a dipole (\(B_0 = 0\) nT) with an initial \(\alpha_{eq} = 90^\circ\). A frequency dependence with \(m_f = -2\) in equation (19) and no \(L\) dependence (\(m_L = 0\) in equation (18)) was assumed. Figure 6a shows results from a simulation done using only an azimuthal electric field with no magnetic perturbation. A least squares fit was applied to the time series of particle spread, and the slope of the best fit line, the red curve in

Figure 3. Magnetic and electric field latitude dependence assumed. Shown are (a) \(B_\phi\) (green curve), \(B_0\) (blue curve), and \(\sqrt{B_\phi^2 + B_0^2}\) (red curve) and (b) \(E_\phi\) (red curve). This is the same as Figure 3 except that \(m_L = 1/3\).

Figure 4. Magnetic and electric field latitude dependence assumed. Shown are (a) \(B_\phi\) (green curve), \(B_0\) (blue curve), and \(\sqrt{B_\phi^2 + B_0^2}\) (red curve) and (b) \(E_\phi\) (red curve). This is the same as Figure 3 except that \(m_L = 1/3\).

Figure 5. Typical daily set of four 3-hour power spectra from the Air Force Geophysics Laboratory magnetometer network, station Camp Douglas, Wisconsin (55° corrected geomagnetic latitude), \(x\) component (magnetic north) for 14 June 1978, \(Ap = 4\), bandwidth = \(2.0 \times 10^{-3}\) Hz, and 42 degrees of freedom [Bloom and Singer, 1995]. The data can be fit by \(f^{-2}\) between 0.1 and 10 mHz.
Figure 6. Mean square deviation of $L$ with respect to time for an ensemble of particles initially at $L = 6.5$, $\alpha_{eq} = 90^\circ$, and $M = 273$ MeV G$^{-1}$ in a dipole magnetic field ($b_c = 0$ nT) with $m_f = 0$ and $m_i = -2$. Blue curves are simulated results, and red curves are best fit lines. Shown are (a) results using only $E_0$ ($B_{pol} = 0$) and (b) results using $E_0$ and $B_{pol}$.

[26] Recall that Figure 6b uses a dipole magnetic field with $M = 273$ MeV G$^{-1}$ and Figure 7b uses a compressed dipole with $M = 460$ MeV G$^{-1}$, both starting at $L = 6.5$ in the equatorial plane ($\alpha_{eq} = 90^\circ$), using fields that are frequency-dependent ($m_i = -2$) and $L$-independent ($m_f = 0$). Figure 6b (dipole) has a $D_{LL} = 0.011$ h$^{-1}$, and Figure 7b has a $D_{LL} = 0.010$ h$^{-1}$. The diffusion coefficient is a sum of contributions from three resonances in the compressed dipole case [Elkington et al., 2003]:

$$D_{LL} = D^{m-1}_{LL} + D^m_{LL} + D^{m+1}_{LL}.$$  

whereas there is only one resonance, $D^m_{LL}$, for a dipole magnetic field. Therefore one might expect the diffusion coefficient to be larger for a compressed dipole (0.010 h$^{-1}$) than for a dipole (0.011 h$^{-1}$) if $D^m_{LL}$ remained the same or was comparable in both cases. In a dipole magnetic field, however, the electrons always drift parallel to the azimuthal electric field given by equation (24). In our compressed dipole model, because the electric field components are defined in a symmetric dipole, the electrons will not be drifting exactly parallel to the azimuthal electric field and will therefore gain less energy over the course of a drift orbit (equation (6)). Therefore, even though $M$ is larger, the diffusion coefficient is about the same for the compressed dipole model. If asymmetric poloidal fields, corresponding to the asymmetry of the magnetic field between noon and midnight, were used, it is expected that the diffusion coefficient would be larger for the compressed case because of the additional resonances.

[21] $D_{LL}$ values calculated here can be compared to the results obtained by Elkington et al. [2003], who obtained $D_{LL} = 0.043$ h$^{-1}$. They used a constant power spectral density versus frequency with an electric field amplitude of 0.1 mV m$^{-1}$ per frequency interval of 0.1 MHz in a compressed dipole with the equivalent $b_c = 15$ nT, starting particles at $L = 6.6$. An azimuthal electric field was used for the model ULF fields, and the compressional component of the ULF oscillation was ignored. Even though the electric field amplitude and the compression are doubled here.

Figure 7b has a mean square deviation of $L$ of 1 per frequency interval of 0.1 mHz in a dipole magnetic field (Figure 6) has a mean square deviation of $L$ of 1 per frequency interval of 0.1 mHz in a dipole magnetic field (equation (18)), still with no compressional dipole was used. The parameter $m_f$ was selected to be 30 nT, chosen to give geosynchronous magnetic fields of nominal values $\sim$120 nT at local noon and $\sim$60 nT at local midnight. For Figure 7a, no frequency dependence ($m_f = 0$ in equation (19)) and no $L$ dependence ($m_i = 0$ in equation (18)) were assumed. The diffusion coefficient for this simulation is $D_{LL} = 0.037$ h$^{-1}$. For Figure 7b a frequency dependence was included this time ($m_i = -2$ in equation (19)), still with no $L$ dependence ($m_f = 0$ in equation (18)) assumed. The diffusion coefficient for this simulation is $D_{LL} = 0.010$ h$^{-1}$.

\[ D_{LL} = \frac{(L - L_0)^2}{2\tau}, \]  

where $L_0$ is the initial position and $L$ is the position after a given amount of time, $\tau$. An electric field amplitude of 0.245 mV m$^{-1}$ per frequency interval of 0.1 MHz is obtained from Figure 2. The diffusion coefficient for this simulation is $D_{LL} = 0.026$ h$^{-1}$. Figure 6b shows a simulation for the same parameters as Figure 6a except that it includes the magnetic perturbation resulting in $D_{LL} = 0.011$ h$^{-1}$. In the equatorial plane, $B_c = 0$ so only a compressional magnetic ULF field is present. Including the compressional ULF magnetic field component, even in the equatorial plane, reduces the diffusion coefficient by a factor of 2. From equation (6), radial transport is due to the electric field in the direction of the drift path and the time derivative of the magnetic field. Since $E_0$ and $\partial B_0/\partial t$ have opposite phases (Figure 1), adding a compressional magnetic field perturbation appears to reduce the effectiveness of $E_0$ in radial transport and therefore radial diffusion.

[19] Plotted in Figure 7 is the mean square deviation of $L$ as a function of time for electrons with an initial $L = 6.5$ and $M = 460$ MeV G$^{-1}$ with an initial $\alpha_{eq} = 90^\circ$. The electric field amplitude is the same as was used in Figure 6, 0.245 mV m$^{-1}$ per frequency interval of 0.1 MHz. This simulation differs from that shown in Figure 6 in that a compressed dipole was used. The parameter $b_c$ in equation (11) was selected to be 30 nT, chosen to give geosynchronous magnetic fields of nominal values $\sim$120 nT at local noon and $\sim$60 nT at local midnight. For Figure 7a, no frequency dependence ($m_f = 0$ in equation (19)) and no $L$ dependence ($m_i = 0$ in equation (18)) were assumed. The diffusion coefficient for this simulation is $D_{LL} = 0.037$ h$^{-1}$. For Figure 7b a frequency dependence was included this time ($m_i = -2$ in equation (19)), still with no $L$ dependence ($m_f = 0$ in equation (18)) assumed. The diffusion coefficient for this simulation is $D_{LL} = 0.010$ h$^{-1}$.
A smaller diffusion coefficient ($D_{LL} = 0.037 \text{ h}^{-1}$) was obtained and can be attributed to the inclusion of a compressional magnetic field perturbation that was not included in the Elkington et al. [2003] model. This conclusion follows from Figure 6.

Results from frequency-dependent fields ($M_f = 0$) (Figure 7b) can also be compared to the result from Elkington et al. [2003] and the results for $M_f = 0$. The diffusion coefficient for the frequency-dependent PSD case is 0.010 h$^{-1}$. This value is smaller than $D_{LL} = 0.037 \text{ h}^{-1}$ for $M_f = 0$ in Figure 7a and $D_{LL} = 0.043 \text{ h}^{-1}$ obtained by Elkington et al. [2003]. A smaller diffusion coefficient when frequency-dependent fields are assumed can be attributed to the different PSD versus frequency profiles assumed here and that assumed by Elkington et al. Particles see a much larger power at the resonant frequency corresponding to 1 MeV at $L = 6.6$ for the simulation done assuming a constant PSD versus frequency. Therefore the diffusion coefficient when $M_f = 0$ is larger.

Simulations were done for $M_f = 0$ and $M_f = -2$, both for $L$-independent fields ($M_L = 0$). Including frequency-dependent PSD has a significant effect on $D_{LL}$. In fact, with $M_f = 0$, $D_{LL}$ gets larger as energy increases whereas the opposite is true for $M_f = -2$. This result is explained by examining the energy density, $U = \epsilon_0 E^2 + B^2/\mu_0$, using $B^2 = B_{pol}^2 + B_{q}^2$ and $E = E_{fr}$, and the drift frequency of the electrons for both cases. As energy increases, $\omega_d$ increases (equation (14)). For $M_f = 0$, $P_E$ is constant, and $P_E$ increases, resulting in $U$ increasing as frequency increases. Therefore, as energy increases, the drift frequency increases, and so does the power, leading to increasing $D_{LL}$ with increasing energy. The opposite is true for $M_f = -2$ where $P_B$ decreases and $P_E$ is constant, resulting in $U$ decreasing as frequency increases. Therefore, as energy increases, the drift frequency increases, and the power decreases, leading to decreasing $D_{LL}$ with increasing energy. The latter corresponds to the frequency dependence seen in ground-based magnetometer measurements (Figure 5). This energy dependence on PSD frequency dependence is evident in Figure 8.

Simulations were also done with particles starting in a dipole magnetic field ($b_c = 0$ nT) at $L = 5$ for a range of energies, from $W_0 = 0.5$ to 3 MeV in 0.5 MeV increments.
The different colored data points correspond to $M = 273$ MeV G$^{-1}$. Different colored data points correspond to different initial pitch angles. The corresponding colored curves represent a best fit power law, $L^n$, where black is $\alpha_{eq} = 90^\circ$, green is $\alpha_{eq} = 79.4^\circ$, red is $\alpha_{eq} = 69.2^\circ$, blue is $\alpha_{eq} = 59.4^\circ$, and pink is $\alpha_{eq} = 50.3^\circ$. Shown are (a) $m_L = 0$ and $m_f = 0$, (b) $m_L = 0$ and $m_f = -2$, (c) $m_L = 1/3$ and $m_f = 0$, and (d) $m_L = 1/3$ and $m_f = -2$. Note that the vertical scales are different for each plot.

Figure 9. $L$ dependence of $D_{LL}$ on a log scale for an ensemble of particles in a dipole magnetic field with $M = 273$ MeV G$^{-1}$. Different colored data points correspond to different initial pitch angles. The corresponding colored curves represent a best fit power law, $L^n$, where black is $\alpha_{eq} = 90^\circ$, green is $\alpha_{eq} = 79.4^\circ$, red is $\alpha_{eq} = 69.2^\circ$, blue is $\alpha_{eq} = 59.4^\circ$, and pink is $\alpha_{eq} = 50.3^\circ$. Shown are (a) $m_L = 0$ and $m_f = 0$, (b) $m_L = 0$ and $m_f = -2$, (c) $m_L = 1/3$ and $m_f = 0$, and (d) $m_L = 1/3$ and $m_f = -2$. Note that the vertical scales are different for each plot.

The different colored data points correspond to $M = 273$ MeV G$^{-1}$ particles starting at different equatorial pitch angles, $\alpha_{eq}$, and are computed for particles starting in a dipole magnetic field ($b_z = 0$ nT in equation (11)). Corresponding colored curves were fit to the simulation data to obtain the $L$ dependence of the diffusion coefficient. Figures 9a and 9b are for simulations using $L$-independent fields, or $m_L = 0$ in equation (18). Figures 9c and 9d are for $m_f = 0$ and $m_f = -2$. Values of $n$, where $D_{LL} \sim L^n$, are shown in Table 1 along with their associated error. An $L^n$ result with $n$ values varying from 4 to 5 was obtained for $m_L = 0$ and $n_f = 0$ simulations (Figure 9a). The value of $n$ depends weakly on the initial $\alpha_{eq}$. An $L^6$ result was obtained for $m_L = 0$ and $m_f = -2$ simulations (Figure 9b) at all initial pitch angles. Figure 9c shows an $L^n$ result with $n$ varying from 15 to 22 for $m_L = 1/3$ and $m_f = 0$ simulations, and Figure 9d shows an $L^n$ result with $n$ varying from 18 to 26 for $m_L = 1/3$ and $m_f = -2$ simulations, depending on the initial $\alpha_{eq}$. Saturation of $D_{LL}$ at the highest $L$ plotted, $L = 6.5$, is due to electron loss at the outer boundary due to the high diffusion rate at high $L$ and therefore is not included in the calculation of the best fit line to the results in Figures 9c and 9d. Including a radial dependence of ULF wave power consistent with observations triples the $L$ dependence of the diffusion coefficient, as seen when comparing results shown in Figures 9a and 9b to Figures 9c and 9d.
Figure 10 shows the equatorial pitch angle dependence of $D_{LL}$. The different colored data points correspond to different initial $L$ values. Corresponding colored curves were fit with a third-order polynomial to the simulation data to indicate the data trend. Figure 10a was obtained from simulations done with particles starting in a dipole magnetic field with $m_L = 0$ and $m_f = -2$ for $M = 273$ MeV G$^{-1}$. There is a weak dependence in the diffusion coefficients for the different values of $\alpha_{eq}$. Figure 10b is similar to Figure 10a except that $m_L = 1/3$ is assumed. When $L$-dependent fields are used, the diffusion coefficients increase with $\alpha_{eq}$. These results can be understood by looking at $|B|$ versus latitude, the red curves in Figures 3a and 4a. For $L$-independent fields, $|B|$ is relatively constant over the range of latitudes $0^\circ - 20^\circ$ ($\alpha_{eq} = 90^\circ - 50^\circ$), leading to the weak $\alpha_{eq}$ dependence of diffusion coefficients at all $L$ values. Whereas for $L$-dependent fields, $|B|$ increases with $\alpha_{eq}$ leading to a larger diffusion coefficient at the equator. The $\alpha_{eq}$ dependence increases with $L$ in both cases, as the ULF wave field amplitude increases relative to the background dipole.

4. Discussion and Conclusions

There has been considerable effort to determine diffusion rates on the basis of model magnetic and electric fields. Early researchers [Cornwall, 1968; Fälthammar, 1965, 1966, 1968; Birmingham, 1969] considering diffusion of trapped energetic particles obtained an $L^6$ dependence of the radial diffusion coefficient, assuming a symmetric dipole magnetic field and a potential electric field with power at frequencies of $\omega = m_\omega$. They arrived at the following diffusion coefficient for $\alpha_{eq} = 90^\circ$ and $m = 1$:

$$D^{(E)}_{LL} = 2 \left( \frac{c}{3 R_E B_0} \right)^2 L^5 P_d(\omega_d),$$

where $P_d(\omega_d)$ contains the factor $T/(1 + (\omega_d T)^2)$ for a step function increase in electric field power with long decay time, $T$, compared to the drift period [Cornwall, 1968; Brautigam and Albert, 2000]. The latter assumption, appropriate for impulsive changes in the convection electric field, with constant first invariant, yields an $L^{10}$ dependence for $\omega_d T \gg 1$, since $\omega_d \sim L^{-2}$ as seen in equation (14) [Schulz and Lanzerotti, 1974]. A particle’s energy and pitch angle only enter the equation via $\omega_d$. In the case of a harmonic ULF wave oscillation one recovers this same $L^6$ dependence in a symmetric dipole for a flat electric field spectrum versus $L$, seen in Figure 9b.

Fälthammar [1968] also calculated a diffusion coefficient for nonrelativistic particles using a magnetic disturbance in a dipole magnetic field with an associated induced electric field. He obtained a diffusion coefficient that scales as $L^{10}$:

$$D^{(B)}_{LL}(W, L, \theta_m) = \Gamma(\theta_m) 2 \pi^2 \left( \frac{S}{7} \right)^2 \frac{R^2 L^{10}}{B_0^2} \nu^2 P_4(\nu),$$

where $\theta_m$ is the mirror colatitude, $\nu = \omega_d/2\pi$ is the drift frequency, and $P_4(\nu)$ is the power spectrum. For $P(\nu)$

<table>
<thead>
<tr>
<th>$\lambda$, deg</th>
<th>$m_L = 0, m_f = 0$</th>
<th>$m_L = 0, m_f = -2$</th>
<th>$m_L = 1/3, m_f = 0$</th>
<th>$m_L = 1/3, m_f = -2$</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td>4.84 ± 0.19</td>
<td>5.91 ± 0.21</td>
<td>14.9 ± 2.0</td>
<td>18.3 ± 3.2</td>
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<td>6.11 ± 0.23</td>
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</tr>
<tr>
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<td>20</td>
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<td>6.17 ± 0.30</td>
<td>21.8 ± 3.4</td>
<td>26.2 ± 3.8</td>
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</table>

Table 1. Comparing $n$ Values, Where $D_{LL} \sim L^n$, for Various Initial Latitudes in a Dipole Magnetic Field for Four Different Cases of $L$-Dependent and Frequency-Dependent Fields

Figure 10. Equatorial pitch angle dependence of $D_{LL}$ for an ensemble of particles in a dipole magnetic field with $M = 273$ MeV G$^{-1}$. Different colored data points correspond to different initial $L$ values. The corresponding colored curves represent a third-order polynomial fit, where red is $L_0 = 4.0$, green is $L_0 = 4.5$, blue is $L_0 = 5.0$, orange is $L_0 = 5.5$, pink is $L_0 = 6.0$, and turquoise is $L_0 = 6.5$. Shown are (a) $m_L = 0$ and $m_f = 0$, (b) $m_L = 0$ and $m_f = -2$, (c) $m_L = 1/3$ and $m_f = 0$, and (d) $m_L = 1/3$ and $m_f = -2$. 9 of 13
the dependence on drift frequency and energy is eliminated. For particles with mirror points more than $\sim 40^\circ$ from the equatorial plane the diffusion coefficient is only a tenth of that of equatorial particles because $\Gamma(\theta_h)$ decreases by 60% as $\theta_h$ decreases from 90° to 70° [Faëthammar, 1965, Figure 1]. For equatorially mirroring particles this result reduces to $D_L^{(B)}(L) = D_0 L^{10}$. Equation (28) indicates that the pitch angle, $L$, and energy dependence of $D_L^{(B)}$ are separable. This means that the pitch angle dependence of $D_L^{(B)}$ can be determined independently of the $L$ dependence. The same is not true for results obtained here. As seen in Figure 10b, the pitch angle dependence cannot be decoupled from the $L$ dependence. Therefore a general statement for the pitch angle dependence similar to Faëthammar [1968] cannot be made for all values of $L$. For comparison, however, from Figure 10b, where frequency- and $L$-dependent fields were used to calculate diffusion coefficients, for $L = 5$, there is about a 90% decrease in $D_L$ as $\theta_h$ decreases from 90° to 70°.

Faëthammar [1968, equation [14]] arrived at this $L^{10}$ dependence by assuming that the $L$ dependence in the equatorial plane magnetic field perturbation scales as $r$ and therefore, by Faraday’s law, his electric field scales as $r^{-1}$ (in spherical coordinates). In this study, it was assumed that $B_{pol} \sim 1/r$ and therefore $E_{pol} \sim 1$ for $m_L = 0$. If the rest of Faëthammar’s calculation is followed to obtain $D_L^{(B)}$ using this $r$ dependence, then a value of $L^{10}$ would be reached instead of the $L^{10.5}$. When $m_f = -2$, the $P_f(\nu)$ term in equation (28) eliminates the $\nu^{-2}$ term, leaving $D_L^{(B)} \sim L^5$. This $L^5$ is what is seen in Figure 9b where $m_1 = 0$ and $m_f = -2$ for all pitch angles. When $m_f = 1/3$, following Faëthammar’s calculation is no longer straightforward because of the square bracket terms in equations (22) and (23), which come from taking the curl of the vector potential, $A_\phi$ (equation (15)). The best that can be predicted is that the $L$ dependence of $D_L^{(B)}$ will be larger than 6 for the field model assumed here when $P_f(\nu)$ is replaced by $P_f(\nu, L)$.

There has also been considerable effort to determine diffusion rates based on observational evidence. An extensive compilation can be found in the studies by Newkirk and Walt [1968], Lanzerotti et al. [1970], Lanzerotti and Morgan [1973], Holzworth and Mozer [1979], and Selesnick et al. [1997]. Lanzerotti and Morgan [1973] and Holzworth and Mozer [1979], for example, assumed that diffusion rates could be described by equations (27) and (28) and used measured magnetic and electric field spectra to obtain the relevant diffusion coefficients. The others calculated diffusion rates without an assumed $L$ dependence. Two of these works [Newkirk and Walt, 1968; Selesnick et al., 1997] made an attempt to determine the radial dependence of outer zone diffusion coefficients based on particle measurements. Newkirk and Walt [1968] found a radial diffusion coefficient dependence of $L^{10.5}$. This result is consistent with standard models of diffusion driven by magnetic fluctuations. Selesnick et al. [1997] examined electron dynamics over a 3-month period in 1996. They assumed that the radial diffusion coefficient took the form $D_{RL} = D_0 L^{4/3}$ and were able to fit observations from three clear electron injections to a time-dependent radial diffusion equation with losses [Schulz and Lanzerotti, 1974], obtaining a coefficient varying as $L^{11.7\pm1.3}$. This result is consistent with those of Elkington et al. [2003] and Y. Fei et al. (manuscript in preparation, 2005).

This trend was also obtained here when $L$-dependent fields were assumed. The $n$ values in the equation $D_L^{(B)} \sim L^n$ from the simulations described in Figure 9d were larger, $\sim L^{18}$ for $\alpha_{eq} = 90^\circ$, than those from empirical studies. However, this difference may be due to the fact that the values chosen for PSD here were for storm time conditions (September 1998), which corresponds to a solar wind velocity $v_{sw} > 500$ km s$^{-1}$ in the Mathie and Mann [2001, Figure 8] study of ULF wave power versus $L$, whereas the $L^{11.7\pm1.3}$ result obtained by Selesnick et al. [1997] was calculated from data over a period of 3 months during solar minimum (1996).

Brautigam and Albert [2000] compared numerical solutions to the radial diffusion equation (equation (7)) to measured electron fluxes. They included a constant loss term, $\Gamma_T$, for model diffusion coefficients and compared results with observations from the Combined Release and Radiation Effects Satellite (CRRES) satellite for the 9 October 1990 geomagnetic storm. CRRES was in a geosynchronous orbit ($6.3 R_E$) at an inclination of 18.2° relative to the equatorial plane with a periapse at 350 km, an apoapse at 36,000 km, and an orbital period of 9.9 hours. CRRES was operational from July 1990 to October 1991. They used measured fluxes at geosynchronous orbit to calculate predicted fluxes (or phase space densities, $f$) for $3.5 \leq L \leq 6$. The diffusion models used parameterized scaling factors, $D_0$ and $D_0$, by $K_p$ where they assume that the radial diffusion coefficient depends on these factors and $L$. For diffusion due to magnetic fluctuations they started with diffusion coefficients determined from magnetic field measurements at $L = 4$ [Lanzerotti and Morgan, 1973] and $L = 6.6$ [Lanzerotti et al., 1978]. These measurements were used to formulate an expression for $D_L^{(B)}(K_p)$ over the continuous interval of $3 \leq 6$. Assuming an $L^{10}$ dependence, Lanzerotti and Morgan [1973] determined $D_0 = L^{-10} D_L^{(B)}$ from ground measurements of ULF magnetic field fluctuations at $L = 4$ [Lanzerotti et al., 1978] determined $D_0 = L^{-10} D_L^{(B)}$ from magnetic field fluctuations at $L = 6.6$. The discrete values of $D_L^{(B)}$ at $L = 4.0$ and 6.6, evaluated using $D_0^{(4.0)}$ and $D_0^{(6.6)}$, respectively, are plotted in Figure 11 (black symbols) for $K_p = 1$—6. Note that the two data points for each $K_p$ value do not exhibit the theoretical $L^{10}$ dependence predicted by the Faëthammar [1965, 1968] model assumed in their individual determinations. For $K_p = 1$ the scaling goes as $L^{1.9}$ while for $K_p = 6$ it is $L^{1.9}$. This difference might be due to the inward motion of the magnetopause (sometimes inside geosynchronous orbit) during high-$K_p$ conditions which is not taken into account in the Faëthammar [1965, 1968] model calculation. Using this information, Brautigam and Albert [2000] used the following expression:

$$D_L^{(B)}(K_p, L) = 10^{0.56K_p-9.325L^{10}},$$

where the values in the exponent were chosen to maximize agreement with the discrete values of $D_L^{(B)}$ empirically determined for $L = 4.0$ and 6.6 for $K_p = 1$—6. More specifically, a $D_L^{(B)}(K_p, L) = 10^{0.56K_p-9.325L^{10}}$ form was chosen, where $a$ and $b$ were determined so as to minimize the sum of the squares of the differences between $D_L^{(B)}(K_p, L)$ and the
Figure 11. \(D_{\parallel\perp}^{mL}\) in d\(^{-1}\) as a function of \(L\) for \(K_p = 1–6\) after Brautigam and Albert [2000]. Discrete values of \(D_{\parallel\perp}^{mL}\) at \(L = 4.0\) are adapted from Lanzerotti and Morgan [1973]; those at \(L = 6.6\) are adapted from Lanzerotti et al. [1978]. Continuous curves of \(D_{\parallel\perp}^{mL}\) are proportional to \(L^{10}\) (equation (29)). The legend associates a symbol (discrete \(D_{\parallel\perp}^{mL}\)) with its corresponding line style (continuous \(D_{\parallel\perp}^{mL}\)) for a given \(K_p\). Plotted over these data are the results from Figure 9 at \(\alpha_{eq} = 90^\circ\). The green curve is for \(m_L = 0\) and \(m_I = 0\), the purple curve for \(m_L = 0\) and \(m_I = -2\), the red curve is for \(m_L = 1/3\) and \(m_I = 0\), and the blue curve is for \(m_L = 1/3\) and \(m_I = -2\).

discrete values of \(D_{\parallel\perp}^{mL}\). Equation (29) is plotted in Figure 11 for the continuous range of \(L = 3–6.6\) for \(K_p = 1–6\) (black curves).

[32] The colored curves in Figure 11, which are from modeled results discussed in section 3, are the results from Figure 9 at \(\alpha_{eq} = 90^\circ\) in d\(^{-1}\). The values of \(D_{\parallel\perp}\) are comparable to those obtained by Brautigam and Albert [2000]. For the two curves that were calculated using \(L\)-independent fields (\(m_I = -2\) (purple) and \(m_I = 0\) (green)) the \(L\) dependence of \(D_{\parallel\perp}\) is smaller than the \(L^{10}\) assumed by Brautigam and Albert [2000]. On the other hand, for the two curves that were calculated using \(L\)-dependent fields (\(m_I = -2\) (blue) and \(m_I = 0\) (red)) the \(L\) dependence of \(D_{\parallel\perp}\) is larger than \(L^{10}\). The blue curve in Figure 3 was obtained using the fields that most closely resemble data, \(m_I = -2\) and \(m_L = 1/3\). An \(L^{18}\) dependence of \(D_{\parallel\perp}\) for this curve at \(\alpha_{eq} = 90^\circ\) was obtained and is much larger than the \(L^{10}\) assumed by Brautigam and Albert [2000], suggesting that their \(L^{10}\) assumption might not have been large enough. This could also explain why the two data points from Lanzerotti and Morgan [1973] and Lanzerotti et al. [1978], based on measurements of ULF wave power which incorporates \(L\) and frequency dependence implicitly, did not exhibit this theoretical value of \(L^{10}\). It can also be seen that these two curves correspond to higher values of \(K_p\) consistent with the power levels chosen for this study selected for storm-time conditions, i.e., \(v_{sw} > 500\) km s\(^{-1}\) from Mathie and Mann [2001, Figure 8].

[33] Brautigam and Albert [2000] used the diffusion coefficients in Figure 11 to calculate phase space densities and electron fluxes for \(M = 100–1000\) MeV G\(^{-1}\). They did this for the 9 October 1990 magnetic storm which exhibits a typical electron flux signature. The radial diffusion equation that they used with a time-dependent outer boundary condition and a \(K_p\)-dependent diffusion coefficient \(D_{LL}\) was solved to determine the phase space density as a function of \(L\) and time at a given first and nonzero second adiabatic invariant throughout the entire 10-day storm period. The agreement between CRRES and their model results throughout the storm recovery phase was strongly dependent on \(M\). For electrons with \(M \leq 314\) MeV G\(^{-1}\) the overall agreement between CRRES and the model results was excellent, indicating that the dynamics observed throughout the outer radiation belt can essentially be explained in terms of variations at the outer (geosynchronous) boundary and \(K_p\)-dependent radial diffusion. For the higher-energy electrons, with \(M \geq 700\) MeV G\(^{-1}\), they found the dynamics to be more complex. Fundamental differences seen between CRRES and model results showed deficiency of the model. One possible source mechanism that could explain this discrepancy is wave-particle heating, which violates first invariant conservation.

[34] The main conclusion from the study by Brautigam and Albert [2000] was that radial diffusion with diffusion rates on the order shown in Figure 11 was adequate to explain the electron fluxes seen by CRRES during a typical storm for lower-energy electrons but not for higher-energy electrons. The values of \(D_{LL}\) obtained from the study described in section 3 are comparable to those from Brautigam and Albert [2000]. This suggests that the same conclusion can be made for the mechanism used here to obtain radial diffusion rates: The values of the diffusion coefficients are large enough to explain the electron fluxes during a typical storm for low-energy electrons with \(M \leq 314\) MeV G\(^{-1}\) but not for the higher-energy electrons with \(M \geq 700\) MeV G\(^{-1}\).

[35] Elkington et al. [2003] suggested that diffusion rates due to \(m \pm 1\) resonant interaction with poloidal mode fields in a compressed dipole can be at least as important as those of previous diffusion calculations; for example, those described by equations (27) and (28) are commonly used in diffusion models of the radiation belts [Beutier and Boscher, 1995; Bourdarie et al., 1996; Brautigam and Albert, 2000]. They also suggest that in the outer zone, power at frequencies corresponding to all three diffusive resonances must be considered in any diffusive description of the radiation belts. They showed that there was as much radial diffusion due to the \((m + 1) \omega_d\) as the \(momega_d\) mode and that the amount due to the \((m - 1) \omega_d\) mode was smaller (factor of 2) but that all three resonances produce linearly additive radial diffusion coefficients of comparable magnitude. What must also be considered, however, is the amount of power at each frequency (equation (26)). If a decreasing PSD is assumed as was done in Figure 7b, then there is more power at the \((m - 1) \omega_d\) mode than at the \(momega_d\) and even less at the \((m + 1) \omega_d\) resonance. This can significantly affect the diffusion rates as demonstrated in the different rates obtained by Elkington et al. [2003] (0.043 h\(^{-1}\)) versus those obtained here (0.037 h\(^{-1}\) for \(m_I = 0\) and 0.010 h\(^{-1}\) for \(m_I = -2\)) for the compressed dipole case shown in
Figure 27. Note that a smaller $M = 460$ MeV $\cdot$ G$^{-1}$ was used here in the compressed dipole case for comparison with Brautigam and Albert [2000], relative to $M = 1890$ MeV $\cdot$ G$^{-1}$ assumed by Elkington et al. [2003]. Even in a dipole magnetic field, effects of the frequency-dependent ULF waves can be seen. Comparing results from Figures 9a and 9c, where $m_L = 0$, to Figures 9b and 9d, where $m_L = -2$, the values of $D_{UL}$ are larger for Figures 9a and 9c which have frequency-independent power. This is because there is more power for any given drift frequency resulting in higher diffusion coefficients.

A first attempt at numerically quantifying the pitch angle dependence of the diffusion coefficient was made by following electron guiding center motion in a background dipole magnetic field, incorporating frequency and $L$-dependent (and $L$-independent) power. It was determined that using the model described here, the radial diffusion rate increases as $\alpha_{up}$ increases for a PSD increasing with $L$ for $m_L = 1/3$. A weak decrease in $D_{UL}$ with increasing $\alpha_{up}$ is obtained for an $L$-independent PSD. For diffusion driven by electric field impulses, Schulz and Lanzerotti [1974, Figure 27] obtained a faster diffusion rate at smaller $\alpha_{up}$, scaling as $[T(y)/D(y)]^2$, where $y = \sin \theta$. They obtained an opposite dependence on $y$ for magnetic fluctuations in a compressed dipole [Schulz and Lanzerotti, 1974, Figure 26] as Fälthammar [1965] found for a dipole (equation (28)).

Examining equation (6), both the magnetic and electric field fluctuations affect radial transport in our model. While $E_0$ and $B_0$ have a maximum at the equator, $B_1$ has a maximum off equator and makes a larger contribution to the wave magnetic field amplitude in equation (6) off equator when there is no radial gradient in PSD (Figure 3) versus the radial gradient case (Figure 4). Ultimately, the pitch angle variation is controlled in our model by the effect of an $L$-dependent PSD on the curl of $A_\phi$; A new term is introduced which strengthens $B_1$ off equator relative to $B_0$ which is maximum at the equator.

It has been shown here that realistic spectral characteristics play a significant role in the rate of diffusion of relativistic electrons via drift resonance with poloidal mode ULF waves. The diffusion coefficient $L$ dependence in a dipole is larger than observationally determined diffusion rates when $L$-dependent spectral power, as expected and observed for ULF waves, is included. The assumption of a power spectral density amplitude $P(\omega) \sim f^{\alpha}$ with $m_f = -2$ was based on ground magnetometer measurements. Since very long wavelength azimuthal mode structure ($m = 2$) was assumed, these measurements should yield an accurate ULF wave power spectrum at the frequencies and wavelengths modeled here. Additional power in the azimuthal electric field component due to impulsive fluctuations in the convection electric field contributes to the power spectrum measured by Holzworth and Mozer [1979], where $m_f = -1.1$ was determined. This additional contribution to radial diffusion, modeled previously [Cornwall, 1968; Fälthammar, 1968; Birmingham, 1969], will further increase the radial diffusion rate when added to that calculated here. Future studies should include the contributions of both ULF wave power and impulsive fluctuations in the convection electric field due to changing solar wind conditions. Global MHD simulations driven by solar wind parameters at the upstream boundary include both types of fluctuations in $E_0$ (and $E_z$ and $B_z$).

Our long-term goal is to implement the 3-D guiding center equations applied here to a simple ULF wave plus background compressed and dipole magnetic field model to MHD fields specified by the coupled models under development by the Center for Integrated Space Weather Modeling [Elkington et al., 2004].

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