This is an upper-level undergraduate course that will cover roughly 4 topics:

1. How magnetized plasmas behave in space.

2. The physics of the Sun’s interior & atmosphere.

3. The ways that the Sun’s magnetic activity extends out through the solar system.

4. The interaction of solar plasma with the “local” environments of the planets.

It’s essentially the “astrophysics of our solar system.”

In the first lecture, we’ll go through the syllabus in detail.
Several things about MATH...

• This isn’t a survey course; it’s mathematical. But what you’ll be expected to learn is really the fundamental physics ideas. The math is just a tool we use.

• Thus, you’ll get a handout with useful mathematical formulae, which you can use for everything, including exams. (Read Wigner quote)

• Still, you should be familiar with much of the math already (thus the course prerequisites) so it won’t be an unnecessary hurdle to catching on to the physics ideas.

• We’ll also do a lot of approximating! Seeing this done may be surprising, if you’re used to problems with exact solutions. For scientists doing research, approximation/assumption is something we do all the time...

It’s an art to figure out

\[
\begin{aligned}
&\text{what to simplify} \\
&\text{what to neglect} \\
&\text{what to flat-out ignore}
\end{aligned}
\]

Hopefully, by seeing me (and in other courses) derive things [under various simplifying assumptions] you’ll start to get a feel for doing it yourself. It takes a while...

\[
= \quad \text{the “exact equality” will often give way to} \\
\approx \quad \text{“is approximately equal to,” or sometimes even} \\
\sim \quad \text{“very roughly equal to” (within an order of magnitude!?)} \\
\propto \quad \text{and sometimes we just care about which quantities are} \\
\quad \text{“proportional to” one another, ignoring normalizing constants.}
\]

• You should bring a calculator to class... there won’t be many in-class exercises that will need them, but it’s hard to predict which days those will be.

(Okay to use an app, but make sure it has scientific notation.)
ENERGY: We know energy is conserved in total, but it can *change form*.

(?) What are the various forms can it take in the Sun & solar system?

- Kinetic energy (carried by particles)
  - Bulk kinetic energy (of an organized gas flow)
  - Thermal energy (internal energy)
- Radiant energy (carried by photons)
- Potential energy (for particles in any kind of a force field)
  - Gravitational potential energy
  - Electric/magnetic potential energy
  - Nuclear binding energy (the **strong** force)
  - (Is there a **weak** force potential energy? Unstable nuclei have it... they’re just aching to release particles that will have “new” kinetic energy!)
- Rest energy ($E = mc^2$)
- Other kinds of aggregate/potential energy in extended objects: springs with tension, “elastic energy” due to deformation.

Strictly speaking a gas’s thermal energy content ≠ HEAT.

Heat is more about physical processes that produce **changes** in energy – either in form, or in location – and usually with an irreversible increase in entropy (randomness).

Later we’ll talk about heating, cooling, and energy transport.
More about kinetic energy of particles: generally, \( E_K = \frac{1}{2}mv^2 \).

Really, the most general (relativistic) way to write it is:

\[
E_K = \sqrt{p^2c^2 + m^2c^4} - mc^2 = mc^2 \left[ \sqrt{1 + \frac{p^2}{m^2c^2}} - 1 \right]
\]

(total) (rest)

For \( v \ll c \),

\[
E_K \approx mc^2 \left[ \left( 1 + \frac{p^2}{2m^2c^2} \right) - 1 \right] \approx \frac{p^2}{2m} \approx \frac{1}{2}mv^2
\]

and we should note that the full vector momentum \( p \) is related to the full vector velocity \( v \) via

\[
p = \frac{mv}{\sqrt{1 - v^2/c^2}}
\]

\[
\begin{cases}
\text{For} \ v \ll c, \ p = mv \\
\text{This is the quantity conserved in elastic collisions.} \\
v \text{ is capped at} \ c, \ \text{but} \ p \ \text{can go to} \ \infty
\end{cases}
\]

Note that if the “flow field” is organized — i.e., dominated by all particle flowing with the \( \sim \)same “fluid velocity” \( u \) — then the bulk kinetic energy (per particle) of the gas is just \( \frac{1}{2}m|u|^2 \).

If the flow field is random (like we know it should be for a hot gas), there is a substantial thermal/internal energy per particle of \( \frac{1}{2}m\langle v^2 \rangle \), where the angle brackets denote the variance (or average of the square) of the speed. It’s \( \propto k_B T \).

More about radiant energy carried by an individual photon:

Since photons have \( m = 0 \),

\[
E_K = pc = h \nu = \frac{hc}{\lambda}
\]
More about potential energy in a gravitational field:

For test particles of mass $m$ in the field of a gravitating body of mass $M_*$, we know that

$$ E_G = -\frac{GM_*m}{r} $$

where $r$ is distance between the 2 bodies. Negative: trapped in the well!

It can also be written $E_G = \frac{1}{2}mV_{esc}^2$ which defines $V_{esc} = \sqrt{\frac{2GM_*}{r}}$

You might be more familiar with it when talking about small differences between “heights” near the surface of a star or planet.

For $r_1 = R_*$ and $r_2 = R_* + h$ (with $h \ll R_*$),

$$ \Delta E_G = \left(-\frac{GM_*m}{r_2}\right) - \left(-\frac{GM_*m}{r_1}\right) = \frac{GM_*m}{r_1} \left(1 - \frac{r_1}{r_2}\right) = \frac{GM_*m}{R_*} \left(\frac{h}{r_2}\right) $$

$$ \approx \frac{m GM_*}{R_*^2} h = mgh. $$

Electric/magnetic potential energy applies to, e.g., electrons that are bound in atoms.

Nuclear binding energy applies to, e.g., protons/neutrons that are bound in nuclei.

Whenever there is any kind of “potential energy well,” conservation of energy demands balance...

- Bound particles may escape if they are given some external source of energy (e.g., photoionization of atoms; photodisintegration of nuclei). [“endothermic” reactions”]

- If a particle “falls” to a lower—more tightly bound—state, then the extra energy may be released (e.g., photon emission via recombination; or nuclear fusion!) [“exothermic” reactions”]
For quantitatively describing energy, there are various units to keep track of. Let’s review those.

In classical physics, one of the first ways we obtained the energy of a particle was by looking at work done by moving it over a given distance:

\[
\text{work } \left[ \text{J} \right] = \left\{ \begin{array}{c} \text{force exerted } \left[ \text{N} \right] \\ \text{length over which it’s pushed } \left[ \text{m} \right] \end{array} \right. \]

Here, we’ll often talk about the energy of a “parcel” of gas or plasma. The parcel has an arbitrary volume \( V \), and contains \( N \) particles.

We’ll make frequent use of the number density \( n = \frac{N}{V} \), and note that \( n \) has units of \( 1/\text{m}^3 \) (inverse volume), since \( N \) is dimensionless.

If each particle \( i \) carries the same amount of kinetic energy \( \tilde{E}_i \), then the parcel’s total energy is

\[
E_{\text{tot}} = \sum_{i=1}^{N} \tilde{E}_i \approx N \tilde{E}_i
\]

and it makes sense to think about \( \tilde{E}_i \) as the mean energy per particle (i.e., \( \langle \tilde{E}_i \rangle = E_{\text{tot}}/N \)). We can define the energy density

\[
U = \frac{E_{\text{tot}}}{V} = n \langle \tilde{E}_i \rangle
\]

The last expression is useful since it doesn’t depend on how big the parcel is (in absolute terms).

Energy density has units of \( \text{[J m}^{-3} \text{]} \), which happens to be the same as the units of pressure (force per unit area):

\[
P = \left[ \frac{\text{N}}{\text{m}^2} \right] = \left[ \frac{\text{J/m}}{\text{m}^2} \right] = \left[ \frac{\text{J}}{\text{m}^3} \right]
\]

We will see why this is important later.
This is related to the **first law of thermodynamics**, which I’m sure you’ve seen before:

\[ d\tilde{Q} = d\tilde{U} + d\tilde{W} \]

Note the tilde’s: each term has units of “energy per particle.”

This law is really a consequence of energy conservation for a gas parcel, and a list (on right-hand side) of the various different forms that an added “piece of energy” (i.e., net heating \(d\tilde{Q}\)) can take.

If the parcel is sitting still, then the added heat can go into:

- heating it up (duh); i.e., increasing its thermal/internal energy by \(d\tilde{U}\), or
- expanding it; i.e., making it do **net work** on its surroundings, with \(d\tilde{W} = PdV\).

(?) **Theoretically, what else could happen to it, if you feed it energy?**

- The whole thing could be accelerated; i.e., set into motion.
- Phase changes! solid \(\rightarrow\) dust \(\rightarrow\) molecules \(\rightarrow\) atomic gas \(\rightarrow\) ionized plasma

Other important energy units:

**Power:** energy emitted (or absorbed) per unit time.

For stars, it’s luminosity: \(L_* = \left[ \frac{J}{s} \right] = \text{[Watts]}\)

**Energy flux:** There are 2 equivalent ways of looking at it:

1. Power emitted per unit area. For a star-like point-source at the origin, its total \(L_*\) is a constant, but the flux \(F\) decreases as inverse square of distance...

   \[ L_* \text{ per unit area:} \]

   \[ F = \frac{L_*}{4\pi r^2} = \left[ \frac{J}{s \ m^2} \right] \]
(2) Transporting a parcel with a known energy density \( U \) with a velocity \( v \)...

\[
F = vU = \left[ \frac{\text{m}}{\text{s}} \right] \left[ \frac{\text{J}}{\text{m}^3} \right] = \left[ \frac{\text{J}}{\text{s} \cdot \text{m}^2} \right]
\]

Lastly, we’ll eventually need to think about **heating rates**, which can be, e.g.,

“power deposited per unit volume” \( Q = \left[ \frac{\text{J}}{\text{s} \cdot \text{m}^3} \right] \)

which tells us how rapidly the **energy density** of a parcel changes in time,

\[
\frac{\partial U}{\partial t} = Q_{\text{heat}} - Q_{\text{cool}}
\]

and we usually just follow one particular form of \( U \) (e.g., thermal energy):

- The net heating rate \( Q_{\text{heat}} \) tells us how rapidly something else converts its energy INTO thermal energy.
- The net cooling rate \( Q_{\text{cool}} \) tells us how rapidly it’s converted OUT of thermal energy into some other form.

Alternately, we sometimes talk about

“power deposited per unit mass” \( \epsilon = \left[ \frac{\text{J}}{\text{s} \cdot \text{kg}} \right] \)

and if there are no changes in time, we can follow how the Sun’s **luminosity** is “built up” as we pile on more and more mass,

\[
\frac{\partial L}{\partial M} = \epsilon \quad \text{or, equivalently,} \quad L = \int \epsilon \ dM .
\]

The two kinds of heating rates are related by \( Q = \rho \epsilon \).
To bring this into the context of the Sun and the solar system, let’s start
telling the story of the changing forms that energy takes on its journey to us.

1. Before the Sun was born, it was a giant interstellar gas cloud. It became
unstable to gravitational collapse, so the formation of the Sun was all
about accumulating gravitational potential energy.

2. We’ll see that, as the new Sun settled into equilibrium, it also converted
part of that gravitational energy into thermal energy (i.e., its core
became hotter and hotter).

3. When $T_{\text{core}}$ reached a threshold value, the motions of H nuclei became so
rapid that they slammed into one another and induced thermonuclear
fusion. Nucleons arrange themselves into more tightly-bound forms (via
the strong-force potential energy).

4. Fusion reactions are exothermic, so they give off photons that carry
radiant energy out through much of the Sun’s interior.

5. Near the solar surface, though, the most efficient way of transporting
energy is no longer radiation. Convection cells form naturally and
transport the energy upwards by a combination of kinetic & thermal
energy (i.e., bulk flow of hot parcels).

6. At the solar photosphere, radiant energy becomes the most efficient way
of getting the energy out. However, there is also still some residual
kinetic energy in the convective “granulation” motions, and magnetic
energy in the field lines that thread the surface.

7. More than 99.99% of the Sun’s power (generated ultimately by fusion)
comes out in the form of radiation. Different parts of the spectrum
eventually interact with different layers of the Earth’s atmosphere:

- **Visible/IR**: Makes it down to the troposphere (cloud layer) and solid
  surface. Photons absorbed by solid matter & re-emitted as
  $\sim$blackbody.
- **Near UV**: Absorbed in stratosphere (ozone layer); excites electrons &
  dissociates molecules (endothermic reactions).
- **Far UV/X-ray**: Absorbed higher up: ionizes the ionosphere.
8. In the chromosphere and corona, there are small amounts of energy in

- **magnetic energy** (transported up by MHD waves or twisted field-line structures that rise up due to buoyancy; “Poynting flux”),
- coronal heating (**thermal energy** due to ???),
- the solar wind (mostly **kinetic energy** of bulk gas flow),
- SEPs (solar energetic particles); lots of **kinetic energy** per particle.

Most of the energy in the interplanetary magnetic field, solar wind, and SEPs interacts with the Earth’s *magnetosphere* far above the actual gaseous atmosphere.

9. Specifically, the solar magnetic field may “reconnect” with the Earth’s magnetic field, and partially annihilate, converting magnetic energy into both **thermal energy** (heating) and **kinetic energy** (of locally generated energetic particles).

10. Some energetic particles in the magnetosphere slam into technological equipment in orbit, drive currents, and build up potential (*voltage*) in conducting or insulating materials. Eventually, this **electric potential energy** may discharge and damage equipment.

11. Some energetic particles in the magnetosphere propagate down to Earth’s polar regions and excite/ionize atoms in the lower ionosphere, then the atoms recombine or de-excite, emitting auroral **photons**.
Vector Calculus Review

Scalars are fine if you just want to measure the magnitude of a quantity.

Vectors are needed if a quantity has a magnitude and a direction (e.g., wind speed in a weather report). Convention: bold \( \mathbf{V} \).

**Note:** If a quantity has a direction ONLY, without a magnitude, we often use a unit vector (with length 1). Convention: hatted lower-case bold \( \hat{n} \).

In physics, both scalars and vectors will tend to be continuous functions of both position and time...

Scalar \( f(x, y, z, t) \); vector \( \mathbf{v}(x, y, z, t) \) is really made up of

\[
\begin{align*}
  v_x(x, y, z, t) \\
v_y(x, y, z, t) \\
v_z(x, y, z, t)
\end{align*}
\]

We should strive to be equally comfortable with manipulating vectors as with scalars...

**Addition:** \( \mathbf{A} + \mathbf{B} \) (the “resultant” effect of following both)

...and the “triangle inequality” says that \( |\mathbf{A} + \mathbf{B}| \leq |\mathbf{A}| + |\mathbf{B}| \)

We’ll often use the shorthand notation that \( A = |\mathbf{A}| \), and so on.

**Multiplication:** the dot/scalar product: \( \mathbf{A} \cdot \mathbf{B} = AB \cos \theta \)
It’s the projection of \( \mathbf{A} \) along \( \mathbf{B} \), and also the projection of \( \mathbf{B} \) along \( \mathbf{A} \).

Overall, it tells you how well 2 vectors line up:

- If \( \mathbf{A} \) is parallel to \( \mathbf{B} \), \( \mathbf{A} \cdot \mathbf{B} = AB \)
- If \( \mathbf{A} \) is perpendicular to \( \mathbf{B} \), \( \mathbf{A} \cdot \mathbf{B} = 0 \)
- And, of course, \( \mathbf{A} \cdot \mathbf{A} = A^2 \)

Pause: We’ll use different coordinate systems, but the most basic one is Cartesian...

\[
\mathbf{A} = A_x \hat{e}_x + A_y \hat{e}_y + A_z \hat{e}_z \quad \text{(sometimes unit vectors given as } i, j, k) \\
\mathbf{A} + \mathbf{B} = (A_x + B_x) \hat{e}_x + (A_y + B_y) \hat{e}_y + (A_z + B_z) \hat{e}_z \\
\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z
\]

For scalars, just 1 type of multiplication is enough. For vectors, we need more!

**Multiplication:** the cross/vector product: \( \mathbf{A} \times \mathbf{B} = (AB \sin \theta) \hat{n} \)

where \( \hat{n} \) is a unit vector perpendicular to both \( \mathbf{A} \) and \( \mathbf{B} \), with a direction formed by the right-hand rule.

The cross product picks out how well 2 vectors are *transverse* to one another:

- If \( \mathbf{A} \) is parallel to \( \mathbf{B} \), \( \mathbf{A} \times \mathbf{B} = 0 \)
- If \( \mathbf{A} \) is perpendicular to \( \mathbf{B} \), \( |\mathbf{A} \times \mathbf{B}| = AB \)
- And, of course, \( \mathbf{A} \times \mathbf{A} = 0 \)
In Cartesian coordinates,
\[
\mathbf{A} \times \mathbf{B} = \begin{vmatrix}
\hat{e}_x & \hat{e}_y & \hat{e}_z \\
A_x & A_y & A_z \\
B_x & B_y & B_z
\end{vmatrix} = \hat{e}_x (A_y B_z - A_z B_y) + \hat{e}_y (A_z B_x - A_x B_z) + \hat{e}_z (A_x B_y - A_y B_x)
\]

In physics, the cross product is useful in cases where quantities have the most effect when they’re NOT lined up – e.g., **torque, magnetic fields**.

The handout gives various kinds of commutative, distributive properties. You don’t have to memorize them, but it’s good to know they exist.

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**Vector Calculus:** Just like multiplication, we need >1 type of derivative to fully describe rates of change of vectors.

We also need vector derivatives to quantify the rates of change of **scalar fields**.

If there’s a scalar function of just one variable \( f(x) \), then its derivative \( df/dx \) (local slope) tells you all you need to know about its rate of change.

What about a scalar function of 2 variables \( f(x, y) \), like elevation as a function of position on the Earth’s surface?

The “slope” through a vertical cut is different depending on the orientation of the cut:

\( (\partial f/\partial x) \) tells us something different from \( (\partial f/\partial y) \).
You’ve seen that the **gradient** describes the full range of slopes in a multi-dimensional scalar field. For a function of all 3 variables $f(x, y, z)$, the gradient

$$\nabla f = \hat{e}_x \frac{\partial f}{\partial x} + \hat{e}_y \frac{\partial f}{\partial y} + \hat{e}_z \frac{\partial f}{\partial z}$$

points in the direction of *maximum increase* of $f$.

(i.e., if $f$ doesn’t vary in the $z$ direction, then $\frac{\partial f}{\partial z} = 0$, and thus the $z$-component of the gradient will have zero contribution to $\nabla f$)

For vector fields, there are 2 types of derivative (analogous to the dot and cross products for multiplication): **divergence** and **curl**.

Divergence measures how much a field spreads out from a given point. Curl measures how much a field swirls or torques around a given point.

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \quad \nabla \times \mathbf{F} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Positive divergence: **expansion** ... Negative divergence: **contraction**.

Watch out, though; vector fields may have *zero* divergence even if the vectors look like they’re diverging from a point. It all depends on how the magnitude varies with distance.
Example: in spherical coordinates, consider a radial vector with $\mathbf{F} = (r^n)\hat{e}_r$. Thus,

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_r) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^{2+n}) = \begin{cases} (2 + n)r^{n-1}, & \text{if } n \neq -2 \\ 0, & \text{if } n = -2 \end{cases}$$

Also, vector fields may have nonzero curl even if the vectors are straight lines! This *shear flow* has a curl:

![Diagram showing shear flow](image)

The handout gives various kinds of identities that boil down to the **chain rule**. You don’t have to memorize them, but it’s good to know they exist.

Just like how dot & cross products “pick out” completely different (mutually orthogonal) projections of a vector, the div and curl (or the grad and curl) “cancel each other out:”

$$\nabla \times \nabla f = 0 \quad \nabla \cdot \nabla \times \mathbf{F} = 0$$

However,

$$\nabla \cdot \nabla f = \nabla^2 f \neq 0 \quad (\text{“Laplacian operator”})$$
**Convective derivative:**

If something is changing in time, we know how to quantify it.

**Example:** Sitting in a meadow, looking at the trees. Let’s follow Cravens, Chapter 2.1.3, and define $Q$ as the density of trees in space.

If we’re just sitting still, the trees are (sloooowly) growing where they stand. Nothing is moving around in space. Thus, $Q = Q(t)$ only, and the total derivative

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t}$$

is just the partial derivative with respect to time.

However, what if we’re in a boat on a river, drifting along the $x$ direction, and the trees are getting denser/thicker as we go down river. In our reference frame, $Q = Q(t, x)$, and $x$ itself is a function of time.

$$Q = Q(t, x(t)), \quad \frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + \frac{dx}{dt} \frac{\partial Q}{\partial x} \quad \text{(chain rule!)}$$

If $Q$ increases as $x$ increases, then $\partial Q/\partial x > 0$. We see the summed effect of 2 kinds of “increase” in tree density: in time, and in space.

In general, if we’re looking at a little “fluid parcel” (that’s evolving in time AND moving around in space with velocity $v$), the total change in some quantity $Q$ associated with the parcel (density, temperature, etc.) is given by

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + v \cdot \nabla Q$$

Remember that $\partial Q/\partial t$ is just the change in the *local* value of $Q$ at a fixed location, as parcels move through it.

But often we want to track what’s going on with the parcel as it moves, and for this we want the above **convective (advective) derivative.**
Coordinate systems

Depending on the symmetries in a system, we might find it more useful to use other coordinate systems besides Cartesian.

I’ll assume you’ve seen cylindrical and spherical before:

It’s straightforward to look up the conversions from \((x, y, z)\) to either \((r, \phi, z)\) or \((r, \theta, \phi)\).

In vector derivatives, extra terms pop up in anything other than Cartesian coordinates, since the unit vectors aren’t fixed in space. See the handout.
Differential Equations

You certainly don’t need to have taken a Diff.Eq course, but you should certainly know how to deal with 1st order separable ODEs.

Example: \[
\frac{dy}{dx} = x^5 y \quad \rightarrow \quad \int \frac{dy}{y} = \int dx \, x^5 \quad \rightarrow \quad \ln y = \frac{x^6}{6} + C.
\]

Sometimes we’ll know the integration limits (i.e., boundary conditions).

For a 1st order linear equation of the form

\[
\frac{dy}{dx} + P(x)y(x) = Q(x)
\]

it’s not separable any more, but a basic technique is the integrating factor

\[
\mu(x) = \exp \left[ \int P(x') \, dx' \right]
\]

which lets us find the solution

\[
y(x) = \frac{1}{\mu(x)} \left[ \int x Q(x') \mu(x') \, dx' + C \right].
\]

Partial differential equations (PDEs) are nastier, but there are a few types that come up again and again in physics:

The advection equation for \( F(x, t) \) is

\[
\frac{\partial F}{\partial t} = -V \frac{\partial F}{\partial x}
\]

and with constant \( V \), it just describes a pattern moving with a known speed \( V \). (It’s essentially \( DF/ Dt = 0 \)... following a parcel!)

If, at \( t = 0 \), it’s satisfied by \( F_0(x) \), then you can show that it’s satisfied at all later times by

\[
F(x, t) = F_0(x - Vt).
\]
We all should be able to recognize a **wave equation** when we see it,

\[
\frac{\partial^2 F}{\partial t^2} = V^2 \frac{\partial^2 F}{\partial x^2}
\]

and it has an oscillating sinusoid type solution like

\[
F(x, t) = F_0 \sin(kx - \omega t) \quad \text{where } V = \omega/k \text{ is the wave phase speed.}
\]

Lastly, the **diffusion equation** is a hybrid: 1st order in time, 2nd order in space:

\[
\frac{\partial F}{\partial t} = D \frac{\partial^2 F}{\partial x^2}
\]

where the diffusion coefficient \(D\) has units of length\(^2\)/time.

If it’s a sharp delta function, \(\propto \delta(x - x_0)\), at \(t = 0\), then at later times, it’s a continuously spreading Gaussian,

\[
F(x, t) \propto \frac{1}{\sqrt{4\pi Dt}} \exp \left[ -\frac{(x - x_0)^2}{4Dt} \right]
\]

which you don’t need to memorize, but just note that the **width** of the Gaussian is roughly given by \(\sqrt{4Dt}\); it increases in time, and it increases faster for larger \(D\).

As \(t \to \infty\), \(F\) diffuses to a constant value. If \(x\) subtends all space, \(F \to 0\).

For many of the above, in 3D one should replace \(\partial/\partial x\) by \(\nabla\).
Electromagnetic Theory Review

Cravens’ Appendix has a brief recap. We’ll use SI units (rusty for me).

Plasmas are not really a “vacuum” because they’re filled with particles. However, we’ll use the vacuum E&M equations and treat the particles as discrete, point-like “add-on” sources of charge.

Thus, forget \( \mathbf{D} \) and \( \mathbf{H} \). Just electric field \( \mathbf{E} \) and magnetic field \( \mathbf{B} \).

If you know \( \mathbf{E} \) and \( \mathbf{B} \), you know the Lorentz force exerted by them on a particle with charge \( q \),

\[
\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})
\]

- Note that a stationary charge only feels the effects of an electric field, and that force is parallel to \( \mathbf{E} \).

- **Motion** is required to feel the effects of magnetic field, and the Lorentz force is perpendicular to both \( \mathbf{B} \) and the particle’s current velocity vector.

\[
\mathbf{E}(r) \text{ and } \mathbf{B}(r) \text{ are continuous vector fields. At each point in space, you can draw a little vector, and then later “connect” them with field lines.}
\]

For \( \mathbf{E} \), field lines = *lines of force*. In absence of other forces, particles will be accelerated parallel to field lines.

Electric field lines begin & end with electric charges.

Magnetic field lines are all CLOSED... there are no magnetic charges.
The magnetic Lorentz force *isn’t* parallel to the field lines, but they’re still useful:

- Some pieces of physics make the most sense when thinking about “how many” field lines cross through a surface.
- Sometimes one can *see them!* Plasma often organizes itself ALONG field lines – i.e., rates of conductivity, particle transport, etc., are quick/easy in the parallel direction; not in the perpendicular direction. *(Coronal loops!)*
- In some ways, we’ll see that field lines have some elasticity/tension to them; i.e., they behave like taut wires.

The combined system of E&M fields and charged particles obeys all four **Maxwell’s equations**, which relate $\mathbf{E}$ & $\mathbf{B}$ to:

- $\rho_c$ (charge density): just what it says: how much electric charge is concentrated into a given volume (Coulombs/m$^3$).
- $\mathbf{J}$ (current density): how much charge is in motion in a given volume (units: $\rho_c \mathbf{v}$... also: Amps/m$^2$). To get total current $I$ (in A) passing through a given surface area $dA$, integrate over $\mathbf{J} \cdot dA$.

In Maxwell’s equations, $\rho_c$ & $\mathbf{J}$ mainly are “sources” for $\mathbf{E}$ & $\mathbf{B}$. However, $\mathbf{E}$ & $\mathbf{B}$ also feed back on $\rho_c$ & $\mathbf{J}$, so the whole system is complex & coupled.

The first of Maxwell’s equations...

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\varepsilon_0} \quad \text{(Gauss’s law)}$$

which tells us how electric fields are *generated* by charge imbalances.

Charge density $\rho_c$ is in units of C/m$^3$.

$\varepsilon_0 = 8.8542 \times 10^{-12}$ (SI); the “permittivity of vacuum.”

We already know this from Coulomb’s law: a point-charge in a vacuum exerts an electric field! The magnitude of this field due to a point charge $q_1$ (at the origin) is

$$|\mathbf{E}| = \frac{k_c |q_1|}{r^2} \quad \text{where Coulomb’s constant} \quad k_c = \frac{1}{4\pi\varepsilon_0}.$$
The associated Lorentz force on a test particle $q_2$, due to the point charge $q_1$ at the origin, has a magnitude

$$|F| = \frac{k_e |q_1 q_2|}{r^2}$$

and the **electrostatic potential energy** due to these two charges is the work done bringing in the test charge from infinity,

$$U_E = -\int_{\infty}^{r} F \cdot dr = \frac{k_e q_1 q_2}{r}$$

which we’ll use later to compare with other kinds of energy.

The 2nd Maxwell’s equation is:

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad \text{ (Faraday’s law)}$$

Another way to generate an electric field is to have a time-varying magnetic field. Faraday invented the *dynamo*.

The last two Maxwell’s equations involve the magnetic field.

$$\nabla \cdot B = 0 \quad \text{ (law of no magnetic monopoles)}$$

Also known as the conservation of magnetic flux. Any closed volume must have equal amount of **field lines** poking INTO it as poking OUT of it.

$$\nabla \times B = \mu_0 \left\{ J + \varepsilon_0 \frac{\partial E}{\partial t} \right\} \quad \text{ (Ampère’s law)}$$

Magnetic fields can be generated by either currents (moving charges relative to one another), or by time-variability of the electric field.

$$\mu_0 = 4 \pi \times 10^{-7} \text{ (SI); the “permeability of vacuum.”}$$

Current density $J$ has units of A/m².

In plasma physics, we will often note that the **displacement current** term $(\partial E/\partial t)$ has a tiny magnitude compared to the other terms, and it will be ignored.
However, for E&M waves propagating in a pure vacuum \((J = 0)\), the displacement current term is important.

One can manipulate Maxwell’s equations into a \textbf{wave equation} for electromagnetic fluctuations.

Take the curl of both sides in the vacuum version of Ampère’s law:
\[
\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \epsilon_0 \left( \nabla \times \frac{\partial \mathbf{E}}{\partial t} \right) = \mu_0 \epsilon_0 \frac{\partial}{\partial t} (\nabla \times \mathbf{E})
\]
and we can use Faraday’s law to rewrite the right-hand side:
\[
\nabla \times (\nabla \times \mathbf{B}) = -\mu_0 \epsilon_0 \frac{\partial}{\partial t} \left( \frac{\partial \mathbf{B}}{\partial t} \right).
\]
Lastly, use a vector identity to rework the left-hand side:
\[
\nabla \times (\nabla \times \mathbf{B}) = \nabla (\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}
\]
and, since \(\nabla \cdot \mathbf{B} = 0\), it leaves us with a wave equation,
\[
\nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \quad \text{with phase speed} \quad V_{\text{ph}} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \equiv c.
\]

We could have started by taking the curl of Faraday, then plugging in Ampère, and we’d get the same wave equation for \(\mathbf{E}\).

Throughout the course, we’ll deal a lot with electromagnetic radiation that passes through gas & plasma. Atoms & ions absorb and/or scatter radiation. (e.g., if \(J \neq 0\), there’d be a first-order \(\partial/\partial t\) in the wave equation: waves would be damped)

In the homework, you’ll show how Maxwell’s equations lead to an expression for electromagnetic energy conservation, with a net “loss” when \(J \neq 0\).

\((?)\): Where does the energy go when currents drain it from E&M fields?

Think about resistors in circuits, or the tungsten elements of old incandescent light bulbs: \textbf{heat!} (& maybe some light)