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The alternative paradigm for magnetospheric physics

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Abstract. This paper emphasizes that the macrodynamics of the terrestrial magnetosphere is more effectively treated in terms of the primary variables **B** and **v** (the **B**, **v** paradigm). The common practice of relating the dynamics to **E** and **j** (the **E**, **j** paradigm) provides direct answers in a variety of symmetric cases, but breaks down in even so simple a static problem as a flux bundle displaced (perhaps by reconnection with the magnetic field in the solar wind) from its normal equilibrium position in a static dipole field. The essential point is that a direct derivation from the equations of Maxwell and Newton leads to field equations written in terms of the continuum fields **B** and **v**. The equations can be recast in terms of **E** and **j**, of course, but they are then unwieldy, being integrodifferential equations. Hence the **E**, **j** paradigm, when correctly applied, is seriously limited in its effectiveness in dynamical problems. Circumventing the limitations with the common declaration that **E** is the prime mover, actively penetrating from the solar wind into the magnetosphere, provides dynamics that is unfortunately at variance with the results that follow directly from Maxwell and Newton. The paper outlines the standard derivation of the basic field equations and then goes on to treat a variety of circumstances to illustrate the effectiveness of the deductive **B**, **v** paradigm in the continuum dynamics of the magnetic field and plasma. There is no attempt to develop a comprehensive model of magnetospheric activity. However we suggest that the ultimate task is more effectively attacked with the **B**, **v** paradigm.

1. Introduction

The theory of magnetospheric activity is commonly expressed in terms of the electric field $\mathbf{E}(\mathbf{r},t)$ and the electric current density $\mathbf{j}(\mathbf{r},t)$, with the view that the electric field drives both the electric current and the bulk plasma motion $\mathbf{u} = c\,\mathbf{E}\times\mathbf{B}/B^2$, while the current causes the magnetic perturbations $\Delta\mathbf{B}(\mathbf{r}, t)$. We refer to this formulation as the $\mathbf{E}$, $\mathbf{j}$ paradigm. The paradigm works from the basic principles of symmetry, charge conservation, the unperturbed field lines of a static magnetic field [cf. *Stern*, 1992], and Ampere's law. The electric field within the magnetosphere is regarded as the active inward extension of the electric field $\mathbf{E}_s = -\mathbf{v}_s \times \mathbf{B}_s/c$ in the solar wind with velocity $\mathbf{v}_s$ and magnetic field $\mathbf{B}_s$. The paradigm provides convenient relations, via the Biot - Savart integral form of Ampere's law, between the electric currents of energetic particles and the associated magnetic field perturbations in static configurations. However the effective application is limited by the fact that the principles of symmetry and equipotential field lines do not apply to nonsymmetric and time-dependent circumstances of magnetospheric activity. Charge conservation combined with the unperturbed field is not always sufficient to provide the essential current paths for computing the field perturbations, even for arbitrarily small static perturbations of a known field configuration. Furthermore, the inward projection of the interplanetary electric field along the magnetic field lines is not applicable unless the convection of the ionosphere is keeping up with the field line convection at the magnetopause so that the field is stationary $(\partial/\partial t = 0)$. So the paradigm, however convenient in simple cases, becomes ineffective for treating strong, nonsymmetric, or time dependent effects.

The purpose of the present paper is to exhibit some of the limitations of the $\mathbf{E}$, $\mathbf{j}$ paradigm with specific examples, and then to elaborate the alternative $\mathbf{B}$, $\mathbf{v}$ (or $\mathbf{B}$, $\mathbf{u}$) paradigm, working with the magnetic field $\mathbf{B}(\mathbf{r}, t)$ and the plasma velocity $\mathbf{u}(\mathbf{r}, t)$ or $\mathbf{v}(\mathbf{r}, t)$ in the dynamical equations [*Parker*, 1962, 1979, 1994; *Siscoe*, 1983]. The essential point is that the magnetospheric system is driven by the internal forces of trapped particles and plasmas and by external forces exerted by the solar wind, rather than by applied macroscopic electric fields. Hence the magnetospheric system is described by the dynamical equations relating momentum and force, that is Newton's equations of motion containing the appropriate Maxwell, Reynolds, and plasma pressure stresses. The Maxwell and Reynolds stresses are expressible directly in terms of $\mathbf{B}$ and $\mathbf{v}$, so that the dynamical equations are naturally formulated in terms of $\mathbf{B}$ and $\mathbf{v}$ and provide a direct deductive approach to magnetospheric physics. As we shall see, no physics is omitted in failing to mention the electric current in treating the deformation of $\mathbf{B}$ by particle forces (or the reverse). The considerations on charge conservation, playing an essential role in the $\mathbf{E}$, $\mathbf{j}$ paradigm, are automatically taken care of by Maxwell's equations (which guarantee charge conservation) and by Newton's equations (which automatically provide ion and electron motions consistent with the electric current requirements of Ampere's law) on which the $\mathbf{B}$, $\mathbf{v}$ paradigm is based.

To state this in other terms, the macroscopic behavior (on scales large compared to the the cyclotron radii of the individual ions and electrons) of the magnetosphere is described by magnetohydrodynamics (MHD) which we refer to here as the $\mathbf{B}$,$\mathbf{v}$

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paradigm. The equations of MHD form a complete set of partial differential equations, providing a deductive approach to the theory of magnetospheric activity. The physical concepts accompanying the MHD equations are mechanical in nature, involving the push and pull of the Maxwell stresses (the pressure and tension of the magnetic field) against the fluid motion, with the field and fluid tied together into a single elastic continuum insofar as the fluid is unable to support an electric field $\mathbf{E}'$ in its own moving frame of reference.

It should be noted that the MHD equations permit surfaces of discontinuity in the circumstances of the planetary magnetosphere, deformed by a quasi-static solar wind. Examples are the magnetopause and auroral sheets. The location of these discontinuities is uniquely defined by the $\mathbf{B},\mathbf{v}$ (MHD) equations applied to the continuous fields in the regions between. On the microscopic scales (the cyclotron radii) the surfaces of discontinuity have finite thickness and internal structure that can be described only by the kinetic equations of plasma physics. The macroscopic condition is simply the balance of the total pressure across the surface of discontinuity. The treatment is analogous to the treatment of a shock transition in a macroscopic hydrodynamic flow, where the location of the shock transition is determined by the large-scale hydrodynamics, with conservation of mass, momentum, and energy (the Rankine-Hugoniot relations) across the small thickness of the shock. The internal structure of the shock can be treated only with application of the complete kinetic theory.

Now the dynamical equations of the $\mathbf{B},\mathbf{v}$ paradigm can, of course, be expressed in terms of $\mathbf{E}$ and $\mathbf{j}$, replacing the bulk plasma velocity by c $\mathbf{E} \times \mathbf{B}/B^2$ and $\mathbf{B}$ by the Biot-Savart integral

$$\mathbf{B}(\mathbf{r}) = \frac{1}{c} \int d^3\mathbf{r}' \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

taken over all space. However, the result is unwieldy, converting partial differential equations into global integro-differential equations. So the $\mathbf{E}$, $\mathbf{j}$ paradigm has no tractable master field equations to guide the theory. Hence in the active convecting magnetosphere the $\mathbf{E}$, $\mathbf{j}$ paradigm is obliged to resort to the ad hoc introduction of individual effects. In the hands of experts the ad hoc construction provides a surprisingly detailed qualitative model of the active magnetosphere [cf. *Fejer*, 1964; *Vasyliunas*, 1970], such as the Rice Convection Model [*Wolf*, 1974, 1983; *Erickson* et al., 1991; *Wolf* et al., 1991; *Yang* et al., 1994]. Unfortunately, the physical explanations become increasingly complicated and indirect in the language of the $\mathbf{E}$, $\mathbf{j}$ paradigm, and the quantitative errors are a stumbling block for more complex situations. A single example suffices to illustrate the problem.

The Birkeland currents $\mathbf{j}_\perp$ are often "explained" by noting that they are associated with the nonvanishing divergence of the gradient and curvature drifts of the ions and electrons of the magnetospheric plasma. The mathematical relations between the Birkeland current density $\mathbf{j}_{||}$ and the plasma presure gradient is then worked out, with the conclusion that the Birkeland currents are caused by the plasma distribution. In another direction, it is sometimes stated that the region 2 Birkeland currents play the important role of shielding the low- latitude magnetosphere from the effects of the magnetic substrom, thereby explaining the observed absence of substrom effects at low latitudes, but in fact these statements are truisms, missing the fundamental point that the region 1 Birkeland currents are induced by the magnetic shear between the polar fields (carried by the solar wind into the antisolar direction) and the surrounding lower-latitude fields, not caught up in the solar wind. That is to say, the region 1 currents follow from Ampere's law in the magnetic field deformed by the tailward transport of flux from the sunward magnetopause. The region 2 Birkeland currents arise in the region of the return flow that is forced by the accumulating magnetic and plasma pressure on the nightside. The return flow is blocked from low latitudes by the extreme adiabatic compression of the particles and plasma that would arise in bundles convected to low latitude [*Zhu*, 1993]. The intensity of the region 2 current at any given latitude follows from Ampere's law as proportional to the gradient in the Maxwell stress necessary to drive the return flow of the massive viscous ionosphere. So the Birkeland currents are located where the mechanics of the magnetospheric convection places them. The curvature and gradient drifts are determined by the mechanical deformation of the magnetosphere, rather than vice versa, and are automatically in compliance with the requirements of the mechanical deformation of the magnetic field.

The advantage of the $\mathbf{B},\mathbf{v}$ paradigm is that it proceeds deductively from the field equations of Newton and Maxwell, automatically steering the investigation along the central path and bypassing the nonproblems and complicated indirect relations between the secondary quantities.

The basic simplicity of the principal variations of the magnetosphere has been noted for sometime [cf. *Sharma* et al., 1993], and references therein) from the low dimensionality ($\sim$ 3) of the observed variations. This consideration, together with a desire for a more deductive, less ad hoc, theory for the observed magnetospheric activity, again suggests the $\mathbf{B},\mathbf{v}$ paradigm with its simple mechanical concepts and convenient field equations. From the point of view of $\mathbf{B},\mathbf{v}$, geomagnetic activity is the mechanical consequence of the turbulent mixing of the magnetopause with the solar wind and magnetic reconnection across the magnetopause. This grabs geomagnetic flux bundles and stretches them back into the goetail at the same time that the pressure of the solar wind compresses the magnetosphere, while the plasma and energetic particles within inflate the magnetosphere. In short the dynamics of the magnetosphere is primarily a shoving match between particles and magnetic field, that is between the Reynolds stress and particle pressure on the one hand and the Maxwell stress on the other. The essential point is that these macroscopic stress fields transcend the internal microscopic details of the plasma and its electric currents. If there is a special need to know the microscopic details, they are readily worked out from the macroscopic fields.

This is not to say that electric currents and parallel electric fields do not play a key role in special situations, for example creating the aurora and in the expansion phase of the magnetic substorm [*Zhu*, 1995]. However those important phenomena are set up by extreme conditions developing in $\mathbf{B}$, and the presence of $\mathbf{E}_{||}$ can be established only by first working out the deformation of $\mathbf{B}$ by the mechanical forces exerted by the particles and plasma. Then if $\mathbf{j}_{||}$ proves to be so large as to require a strong $\mathbf{E}_{||}$, the consequences of $\mathbf{E}_{||}$ [*Coroniti and Kennel*, 1973; *Schindler* et al., 1991] must be introduced into the calculation of $\mathbf{B}$. It is not possible to pursue the physics in the opposite order except where observation provides the solution to the problem ahead of time. Further commentary on $\mathbf{E}_{||}$ is to be found in section 3.

The exposition begins with a brief review of some specific limitations of the $\mathbf{E}$, $\mathbf{j}$ paradigm, not always appreciated along side the effective applications of the paradigm. Then the well known dynamical equations are worked out again from Maxwell's equations to illustrate the nature of the $\mathbf{B}$, $\mathbf{v}$ paradigm. Several static magnetospheric phenomena are treated with both paradigms in section 4, from which the reader can see the individual merits and limitations under simple circumstances. Finally, the paper treats some time dependent cases in sections 7 and 10, easily handled with $\mathbf{B}$ and $\mathbf{v}$ but difficult with $\mathbf{E}$ and $\mathbf{j}$. The pitfalls of declaring an equivalent electric circuit for the $\mathbf{E}$, $\mathbf{j}$ paradigm are noted in sections 10 and 11. Section 8 develops the $\mathbf{B}$, $\mathbf{v}$ paradigm in a partially ionized gas appropriate for the ionosphere.

2. The E, j Paradigm

As already noted, the $\mathbf{E}$, $\mathbf{j}$ paradigm considers the electric field $\mathbf{E}$ within the magnetosphere to be a consequence of the active inward penetration of the electric field $\mathbf{E}_s = -\mathbf{v}_s \times \mathbf{B}_s/c$ in the solar wind, following the ideas of *Alfven* [1939, 1950, 1981], *Dungey* [1958, 1961], *Oguti* [1971], *Heikkila* [1974], *Banks* [1979], *Lysak* [1990 and ref. therein]. The conventional $\mathbf{E}$, $\mathbf{j}$ paradigm is stated concisely by *McPherron* [1991, p. 618] in a recent review of substorms.

> ...Since field lines are generally equipotentials, the electric field of the solar wind is transmitted into the magnetosphere and ionosphere by the connected magnetic field lines. This electric field sets up a convection system that moves magnetospheric plasma from the tail behind the Earth to the day side. The ionospheric plasma undergoes the same motion and in addition conducts electrical currents driven by the electric field...

On the other hand, both the $\mathbf{E}$, $\mathbf{j}$ and $\mathbf{B}$, $\mathbf{v}$ paradigms are sometimes indicated in the contemporary literature on magnetospheric activity. We note [*Gonzalez* et al., 1994 p. 5774] where they state that

> ...The primary causes of geomagnetic storms at Earth are strong dawn-to-dusk electric fields associated with the passage of southward directed interplanetary magnetic fields. The solar wind energy transfer mechanism is magnetic reconnection between the IMF and the Earth's magnetic field. The basic energy transfer process in Earth's magnetosphere is the conversion of directed mechanical energy from the flow of the solar wind into magnetic energy stored in the magnetotail, followed by its conversion into primarily thermal mechanical energy in the plasma sheet, auroral particles, ring current, and Joule heating of the ionosphere. Extraction of energy from the solar wind requires a net force between the solar wind and Earth, with force times solar wind speed giving the energy input rate...

Another point to be noted is that the basic role ascribed to $\mathbf{j}$ in the $\mathbf{E}$, $\mathbf{j}$ paradigm, together with the notion of an applied electric field $\mathbf{E}_s$, has led to the idea [*Alfven*, 1981; *Heikkila*, 1974] that magnetospheric activity can be represented comprehensively and precisely by simple electric circuits. The notion is a curious one, that a dynamical system with infinitely many degrees of freedom can be represented by a fixed electric circuit with only a few loops (a few degrees of freedom). In fact, it is often possible to provide an approximate circuit analog once the basic mechanics is worked out from the dynamical equations. However unfortunately, the circuit analog can be established only after the fact, and as noted in section 3, it ignores important aspects of the fluid motions, so it is of little practical use in most cases.

Now the field lines [*Stern*, 1994] are equipotentials only in precisely stationary $(\partial/\partial t = 0)$ conditions [*Stern*, 1977]. The concept of equipotential field lines can be applied to magneto spheric convection [*Gold*, 1959; *Dungey*, 1961; *Axford and Hines*, 1961; *Kellogg*, 1962; *Walbridge*, 1967; *Coroniti and Kennel*, 1973] only in the special case that the ionospheric winds are keeping up precisely with the tailward transport of the individual flux bundles involved in the flux transfer events. However $\mathbf{E}_s$ has no power to maintain this stationary $(\partial/\partial t = 0)$ condition. For instance, when there is no reconnection of field lines between the magnetosphere and the solar wind, the magnetopause is precisely an equipotential surface so far as the external $\mathbf{E}_s$ is concerned. So there can be no electrostatic stimulation of the interior, that is $\mathbf{E}_s$ by itself does nothing to create convection. What little convection there may be ($\sim 10^2$m) is driven by the small–scale fluctuations in the magnetopause [*Axford*, 1962; *Parker*, 1958, 1967a, b, 1969a; *Lerche*, 1966, 1967; *Lerche and Parker*, 1967; *Eviatar and Wolfe*, 1968; *Vasyliunas*, 1970; *Schieldge and Siscoe*, 1971; *Coroniti and Kennel*, 1973; *Tsurutani* et al., 1992].

That is to say, it is the Maxwell stress provided by a nonvanishing perpendicular magnetic field $\mathbf{B}_\perp$ that transmits the momentum of the solar wind across the magnetopause into the magnetosphere. The same $\mathbf{B}_\perp$ provides a potential difference across the magnetic field, but the stress transmitted by the electric field is small $O(v^2/c^2)$ compared to the magnetic stress. *Tsurutani and Gonzalez*, [1995] estimate that the shear stress exerted by the passing solar wind via the small scale fluctuations is approximately $1-3\times10^{-2}$ of the force arising from outright reconnection of the geomagnetic field lines with a southward interplanetary magnetic field at the sunward magnetopause. It is this substantial reconnection and the associated large Maxwell tangential force on the magnetosphere that initiates the substorm, of course.

Imagine, then, that a static magnetosphere is disturbed by the abrupt reconnection of a geomagnetic flux bundle into a southward interplanetary magnetic field $\mathbf{B}_s$ at the sunward magnetopause. The reconnected flux bundle is rapidly transported by the solar wind into the geomagnetic tail, during which time the massive ionospheric footpoint of the reconnected flux bundle responds but little. It follows that the electric field $\mathbf{E} = -\mathbf{v}\times\mathbf{B}/c$ within the reconnected flux bundle declines from $\mathbf{E}_s$ at the magnetopause to zero at the unmoving ionosphere. So the electric field generally does not map downward along the field lines in the active magnetosphere, that is the electric field has no power of penetration to drive the motion of the plasma, contrary to the common assertion. Rather, it is the Maxwell stress exerted on the ionosphere by the tension and pressure of the displaced reconnected magnetic flux bundle that drives the ionosphere. More simply, it is the accumulated displacement, rather than the rate of displacement, of the flux bundle that drives the ionosphere. The electric field maps along the magnetic field lines only if and when the Maxwell stresses can bring the ionospheric convection into a steady state that keeps up with the flux transfer at the magnetopause.

To consider the problem from a different point of view, $\mathbf{E}$, given by $-\mathbf{v}\times\mathbf{B}/c$, cannot be the compelling physical effect, that is the prime mover, because the dynamical equations must be covariant. That is to say, the equations must be couched in terms of physical concepts that are applicable in every inertial frame of reference, for example the frame of the solar wind and the frame of the magnetosphere. So if $\mathbf{E}_s$ were to be considered the prime mover for exciting magnetospheric activity in the frame of reference of Earth, how would we treat the excitation of the passing solar wind by the nonconvecting magnetosphere? There is no significant electric field within the magnetosphere. Hence there would be no prime mover and hence no excitation of the solar wind. On the other hand, if the calculation were carried out in the frame of the solar wind, in which there is then no electric field, there is no basis for activating the magnetosphere. However, there would be an electric field $\mathbf{E}_m = +v_s \times \mathbf{B}/c$ in the magnetosphere which activates the solar wind, contrary to the conclusion reached in the frame of reference of the magnetosphere.

It should not go unnoticed that, with the nonrelativistic Lorentz transformation $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}/c$, an electric field $\mathbf{E}_{\|}$ parallel to the magnetic field, arising from intense field aligned current $\mathbf{j}_{\|}$, is uniquely defined in all frames of reference, $\mathbf{E}'_{\|} = \mathbf{E}_{\|}$, just as $\mathbf{B}$ is well defined and invariant, neglecting terms second order in v/c.

To treat the other half of the $\mathbf{E}, \mathbf{j}$ paradigm consider the often stated concept that $\mathbf{j}$ is the cause of $\Delta\mathbf{B}$. In the usual meaning of the word "cause" this implies that the energy that goes into the creation of $\Delta\mathbf{B}$, increasing the magnetic energy by $\mathbf{B} \cdot \Delta\mathbf{B}/4\pi$ per unit volume, is supplied by $\mathbf{j}$. However, this is not how $\Delta\mathbf{B}$ comes about. The energy source is primarily the kinetic energy of the solar wind which impacts the magnetopause, grabs bundles of flux, and deforms the elastic terrestrial magnetosphere. The fluctuating solar wind and the associated varying deformation provide a varying $\nabla \times \mathbf{B}$, with which there is associated incidentally a varying $\mathbf{j}$, according to Ampere's law. The Maxwell equation $\partial\mathbf{E}/\partial t = c\nabla \times \mathbf{B} - 4\pi\mathbf{j}$ tells us that any momentary deviation of $4\pi\mathbf{j}$ from $c\nabla \times \mathbf{B}$ quickly produces an $\mathbf{E}$ that compels $\mathbf{j}$ through Newton's equations for the motion of the electrons and ions to become equal to $c\nabla \times \mathbf{B}/4\pi$. The electric field is also described by Faraday's induction equation $\nabla \times \mathbf{E} = -(1/c)\partial\mathbf{B}/\partial t$, of course. The energy to create the necessary electric current comes from the deformation of the magnetic field, as is easily shown from Poynting's theorem (section 3). So $\Delta\mathbf{B}$ is the energy source, driven by the mechanical forces of the solar wind, that creates $\mathbf{j}$. That is to say, in the magnetosphere the magnetic field is the cause of the current.

It should be recognized that the situation is different in the laboratory plasma, where the action is often initiated by discharging a condenser bank through the plasma and/or through external current coils. There is one unique reference frame, defined by the laboratory boundaries of the plasma. An enormous potential difference is applied to the boundary of the plasma, in contrast with the equipotential magnetopause. The energy is delivered to the system by the applied and well defined $\mathbf{E}$, causing electric currents to flow and doing work at the rate $\mathbf{j} \cdot \mathbf{E}$. The magnetic fields are caused by $\mathbf{j}$, through Amperes law (i.e., the Biot-Savart integral), and $\mathbf{E}$ is clearly the prime mover.

It is interesting to note that the $\mathbf{E}, \mathbf{j}$ paradigm is also an essential part of the picture, along with $\mathbf{B}$ and the dynamics of the individual ions and electrons, in the plasma structure of the ideal stationary magnetopause over the cyclotron radius of the impinging solar wind ions. The magnetohydrodynamic $\mathbf{B},\mathbf{v}$ paradigm is not applicable to such small scales, of course, and the problem is nonlocal because the impinging solar wind ions play the role of an initial applied emf, driving currents along the field lines into the ionosphere as an essential part of the stationary equilibrium of the magnetopause. Detailed analysis [*Parker*, 1967a, b; *Lerche*, 1967; *Lerche and Parker*, 1967] shows that the ionospheric resistivity dissipates the currents so that there can be no enduring equilibrium. *Eviatar and Wolf* [1968] suggest that the microstructure of the magnetopause is dynamically unstable so that the absence of stationary equilibrium is not confronted in nature, leaving some interesting questions unanswered.

The unipolar inductor provided by the motion of Io relative to the rotating magnetosphere of Jupiter is another example of the proper use of the $\mathbf{E}, \mathbf{j}$ paradigm, with the precisely defined motion of Io providing the driving emf [*Goldreich and Lyndon-Bell*, 1969].

On the other hand, it will become clear in the illustrative examples that the $\mathbf{E}, \mathbf{j}$ paradigm is often hampered by (1) the need to know the perturbed field $\mathbf{B} + \Delta\,\mathbf{B}$ in order to describe the parallel current $\mathbf{j}_{\|}$ to a sufficient degree of precision and (2) the need to know the fluid motion $\mathbf{v}$ before $\mathbf{E}$ and $\mathbf{j}$ can be related by an algebraic (tensor) Ohm's law. Thus it is relatively easy to compute $\mathbf{E}$ and $\mathbf{j}$ once $\mathbf{B}$ and $\mathbf{v}$ are worked out from the dynamical equations (which generally do not depend on $\mathbf{E}$ and $\mathbf{j}$), but it is difficult to go the other way. Unfortunately, these facts are sometimes overlooked and the path of $\mathbf{j}_{\|}$ is based on the unperturbed $\mathbf{B}$, while Ohm's law is applied to $\mathbf{E}$ in the reference frame of the coordinates rather than to the field $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}/c$ in the local frame of the moving fluid. In fact, $\mathbf{E}'$ cannot be computed until after $\mathbf{v}$ and $\mathbf{B}$ have been determined. Thus for instance, the comprehensive magnetospheric models of *Vasyliunas* [1970] and *Wolf* [1983] illustrate how far the $\mathbf{E}, \mathbf{j}$ paradigm can be carried to provide an ad hoc first order representation of geomagnetic disturbance. However, they can provide the correct ionospheric current pattern only in the second iteration with $\mathbf{v}$ deduced from the Lorentz force calculated the first time around. A skillful theoretician can accomplish these iterations, but why use proxy variables and a tricky ad hoc approach when the $\mathbf{B},\mathbf{v}$ paradigm provides the partial differential field equatons for the comprehensive deductive approach? Sections 6.2 and 9 provide illustrations of the difficulties (1) and (2), respectively.

The same general problem arises in the popular practice of declaring an equivalent electric circuit for a dynamical system, forgetting that the current paths are generally not known until the dynamical problem is solved and $\mathbf{B}$ is known and forgetting that the effective electromotive force depends upon the motion of the current path across the magnetic field, which is not known until $\mathbf{v}$ has been determined. In fact, the correct electric circuit is distributed over $\mathbf{v}(\mathbf{r}, t)$, and its proper description is precisely stated by the dynamical equations for $\mathbf{B}$ and $\mathbf{v}$, from which $\mathbf{E}$ and $\mathbf{j}$ are easily computed once $\mathbf{B}$ and $\mathbf{v}$ are determined. Section 11 provides a simple illustrative example.

3. The B, v Paradigm

Application of the $\mathbf{B}, \mathbf{v}$ paradigm to the active planetary magnetosphere [*Parker*, 1962] is based on Newton's equations of motion, including the Maxwell stress, and on Maxwell's equations, along the lines indicated in the general development of the paradigm by *Lundquist* [1952] and *Elsasser* [1954]. The mutual consistency of the equations becomes evident upon close inspection. For instance, Maxwell's equations assert that electric charge is conserved, which is tantamount to asserting conservation of particles, and hence mass. Then Newton's equations can be applied to the motions of the individual ions and electrons of a collisionless plasma to compute the mean electric current density. Substituting the resulting expression for the current density into Maxwell's equation

$$\frac{\partial \mathbf{E}}{\partial t} = c\nabla \times \mathbf{B} - 4\pi\mathbf{j} \tag{1}$$

yields Newton's equation of motion for the macroscopic bulk motion of the plasma [*Parker*, 1957]. Thus the particle dynamics is such as to fulfill Ampere's law automatically. This detailed result, which is described in the next section, is implied by Poynting's theorem, of course, showing that the concept of energy and momentum carries over from mechanics into electromagnetism.

In summary, Newton's equations at the microscopic scale of the individual ions and electrons automatically prescribe motions that produce the electric currents required by Ampere's law. Combining this fact with Poynting's theorem, it follows that the mean macroscopic bulk fluid motion of the ions and electrons is also described by Newton's equations in terms of the mean macroscopic Maxwell stress tensor. This should come as no surprise in view of the general compatibility of classical mechanics and electromagnetic fields in which the equivalent momentum of a particle with charge e can be written $p_i + eA_i/c$, where A_i is the vector potential in the Lorentz gauge. Note, then, that if Ampere's law were not automatically satisfied in the large scale by the individual particle motions, it would not be possible to write the Maxwell stress in terms of $\mathbf{E}$ and $\mathbf{B}$ alone. Some functional of the plasma velocity $\mathbf{v}$ and its derivatives would appear in the Maxwell's stress. Physics would then lose its covariance and we would live in a different world.

It follows that the time dependent macroscopic physics of magnetospheric activity reduces to the dynamics of the contending particle pressure p_{ij} (thermal momentum flux), the Maxwell stress tensor M_{ij},

$$M_{ij} = -\delta_{ij}\frac{B^2}{8\pi} + \frac{B_iB_j}{4\pi}, \tag{2}$$

and the Reynolds stress tensor R_{ij},

$$R_{ij} = -\rho u_i u_j. \tag{3}$$

Thus for a plasma with density ρ and macroscopic bulk motion u_i, the dynamical equation has the basic Newtonian form

$$\frac{\partial}{\partial t}\rho u_i = -\frac{\partial p_{ij}}{\partial x_j} + \frac{\partial R_{ij}}{\partial x_j} + \frac{\partial M_{ij}}{\partial x_j} \tag{4}$$

to which may be added a gravitational force, viscosity, etc. From the fact that $\mathbf{E} = -\mathbf{u} \times \mathbf{B}/c$ in a collisionless plasma it follows from Faraday's law of induction that

$$\frac{\partial B_i}{\partial t} = \frac{\partial}{\partial x_j}(B_j u_i - B_i u_j) \tag{5}$$

which decrees that B_i is transported bodily with u_i.

There are occasional and important exceptions, of course, as already noted. The most interesting addition to the induction equation is a significant electric field $\mathbf{E}_{\parallel}$ parallel to $\mathbf{B}$, arising where the plasma is too tenuous to carry the electric current demanded by Ampere's law. A case in point arises in the region 1 Birkeland currents in the extremely low density ($\lesssim 1$ electron/cm^3) beyond the plasmapause. The appearance of the term $-c\nabla \times \mathbf{E}_{\parallel}$ on the right-hand side of (5) violates the frozen in condition on the field [*Coroniti and Kennel*, 1973; *Schindler* et al., 1991]. If, on the other hand, the plasma, or an un-ionized gaseous background, becomes so dense that there are many interparticle collisions, as is the case in the ionosphere, a resistive diffusion tensor η_{ij} is included on the right-hand side of (5), which also gets away from the frozen in condition.

With these points in mind a variety of magnetospheric phenomena can be understood in simple terms from the disturbed conditions in interplanetary space [*Tsurutani* et al., 1990; *Tsurutani and Gonzalez*, 1993]. For instance, the expansion phase of a substorm appears to be the direct consequence of switching on the same $\mathbf{E}_{\parallel}$ that causes the auroral enhancement [*Zhu*, 1995]. That is to say, the region 1 Birkeland current sheet is a consequence of the tailward displacement of magnetic flux, and when the current density becomes large enough to generate plasma turbulence (anomalous resistivity and electric double layers) the required $\mathbf{E}_{\parallel}$ becomes large. Calculations show that the $\mathbf{E}_{\parallel}$ releases the magnetic field in the outer magnetosphere from its line tying to the ionosphere, thereby facilitating the expansion of the geotail. The sudden release appears to be the cause of the Pi 2 pulsations (Zhu, 1995).

The direct approach of the $\mathbf{u}$, $\mathbf{B}$ paradigm to the driving forces, p_{ij}, R_{ij}, M_{ij}, etc. simplifies many aspects of the theory of magnetospheric activity. The first point is that the current $\mathbf{j}$ flows across $\mathbf{B}$ where, and only where, the plasma pushes against the field. Where the plasma does not push against the field, $\mathbf{j}$ is necessarily parallel to $\mathbf{B}$. So it is the plasma and particle pressure distribution that determines the electric current patterns, which cannot otherwise be constructed. Hence the assertion of an equivalent electric circuit generally cannot be made until the plasma pressure and Maxwell stress have been established. That means a proper dynamical solution to the problem must be in hand. For instance, the region 1 and 2 Birkeland currents follow as a direct consequence of Ampere's law applied to the magnetic flux bundles displaced by the solar wind from the sunward magnetopause into the geomagnetic tail and to the ensuing sunward return flow of field around the periphery of the polar ionosphere [*Zhu*, 1993].

Quantitatively, the tilt of the polar magnetic field B_p (~ 0.6 G) connecting into the geotail is observed above the ionosphere as the change ΔB in the sunward horizontal component, typically -2×10^{-3} G (200 nT). The Maxwell stress $B_p\Delta B/4\pi$ in the antisolar direction on the polar ionosphere is therefore of the order of 10^{-4} dynes/cm^2, initiating ionospheric and magnetospheric convection in the antisolar direction. Typical convective velocities v in the F layer are 1 km/s in the antisolar direction [*Heppner*, 1977], representing the magnetospheric convection above the polar ionosphere. The Maxwell stress does work on the ionosphere at the rate $vB_p\Delta B/4\pi \sim 10$ ergs cm^{-2}s^{-1}. The total work over the polar ionosphere (with a radius of 20^o latitude or about 2×10^8cm) is then of the order of 10^{18}ergs/s $= 10^{11}$ W [*Zhu*, 1994a, b]. This power input is ultimately consumed by viscous dissipation and resistive dissipation, of course (see detailed description in section 11). Zhu points out that the return (sunward) magnetospheric convective flow is driven by the sunward tilted field $\Delta B \sim 8 \times 10^{-3}$ G (800 nT) observed around the periphery of the polar ionosphere. The ionosphere around the periphery is driven sunward by the magnetospheric convection at speeds of the order of 1 km/s, so the power input is of the order of 30 ergs cm^{-2} s^{-1}, over a characteristic width of 10^8 cm, so that the work done is of the general order of 3×10^{18}ergs/s $= 3 \times 10^{11}$ W.

Application of Ampere's law to the boundary between the antisolar (polar) magnetic tilt and the peripheral solar magnetic tilt provides the region 1 intense Birkeland current sheet, while the gradually declining solar tilt beyond the periphery provides the more diffuse region 2 Birkeland current, already noted. The Birkeland currents are a simple and direct consequence of Ampere's law and the tailward transport of magnetic flux bundles that have reconnected with a southward interplanetary magnetic field at the sunward magnetopause.

Finally, it should be noted that the variables $\mathbf{E}$ and $\mathbf{j}$ are easily computed, when needed, from $\mathbf{u}$ and $\mathbf{B}$, with the well known relations $\mathbf{E}_\perp = -\mathbf{u} \times \mathbf{B}/c$ and $\mathbf{j} = c\nabla\times \mathbf{B}/4\pi$.

4. Newton's and Maxwell's Equations

Having asserted the compatibility of Newton's and Maxwell's equations at both the microscopic particle level and at the macroscopic fluid level, it is important to understand the theoretical basis for this salutary condition. First of all, Maxwell's equation (1) implies charge conservation, as already noted. The divergence of (1), with the additional relation $\nabla \cdot \mathbf{E} = 4\pi\delta$, where δ is the charge density, yields

$$\frac{\partial \delta}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

which is the statement of conservation of charge. Second, if the electric current density is inadequate to satisfy Ampere's law $\mathbf{j} = c\nabla\times \mathbf{B}/4\pi$ by some small amount $\Delta\, \mathbf{j}$, it follows from (1) that

$$\frac{\partial \mathbf{E}}{\partial t} = 4\pi\Delta\mathbf{j}$$

The result is a rapid growth of $\mathbf{E}$ in the direction of the inadequacy $\Delta\mathbf{j}$. The current carrying particles behave in a manner described by Newton's equations, so they are accelerated by the increment in $\mathbf{E}$ in such a way as to make up the shortfall in the electric current. That is to say, the existence of the displacement current $\partial\mathbf{E}/\partial t$ guarantees that the electric current follows Ampere's law very closely, with $\partial\mathbf{E}/\partial t$ generally small to second order in v/c compared to $c\nabla \times \mathbf{B}$ and $4\pi\mathbf{j}$. It is this small $\partial\mathbf{E}/\partial t$ that produces the parallel electric field $\mathbf{E}_{||}$ responsible for the aurora. Thus, for instance, in the absence of the current required by Ampere's law a parallel electric field of 10^{-2} volts/m, providing an accelerating potential of 10^4 volts over 10^3km, would arise in 10^{-8} s in a magnetic shear where the direction of $\mathbf{B}$ (0.6 G) changes by 10^{-3} radian in a distance of 2 km across $\mathbf{B}(|\nabla\times \mathbf{B}| = 3\times 10^{-9}$ G/cm). So for all of its usual diminutive size, $\partial\mathbf{E}/\partial t$ has profound effects, providing the aurora and preserving Ampere's law because the particles move according to Newton's equations.

The intimate relation between Newton's and Maxwell's equations can be shown in a variety of ways. For instance, Newton's equations can be deduced from Maxwell's equation (1), while the magnetic induction equation (5) follows from the motion of the particles (in whose frame $\mathbf{E}' = 0$), i.e., from Newton's equations. Briefly, *Watson* [1956] and *Brueckner and Watson* (1956) computed the ion and electron trajectories in a magnetic field $\mathbf{B}$ with characteristic scale ℓ large compared to the cyclotron radii of the individual particles. They used the guiding center approximation and went on to solve the collisionless Boltzmann equation for small perturbations using the computed particle trajectories, which constitute the characteristics of the Boltzmann equation. The result is the equivalent equation of motion for the electric drift velocity $\mathbf{u} = c\,\mathbf{E} \times \mathbf{B}/B^2$ of the particles in the form of (4). In particular there is a contribution from p_{ij} arising from an anisotropic thermal velocity distribution in the presence of the curvature $\mathbf{K}$ of the field lines. The curvature is

$$\mathbf{K} = \frac{[(\mathbf{B}\cdot\nabla)\mathbf{B}]_\perp}{B^2} \tag{6}$$

where the subscript $\perp$ denotes the component perpendicular to $\mathbf{B}$. The plasma pressures $p_\perp$ and $p_{||}$ (perpendicular and parallel to $\mathbf{B}$, respectively) are not always equal in a collisionless plasma, in which case there is a centrifugal force term $\mathbf{K}\,(p_{||} - p_\perp)$ as a consequence of the motion of particles along the curved field lines. The result is the equation of motion

$$\rho\frac{d\mathbf{u}}{dt} = -\nabla_\perp(p_\perp + \frac{B^2}{8\pi}) + \frac{[(\mathbf{B}\cdot\nabla)\mathbf{B}]_\perp}{4\pi}\left(1+\frac{p_\perp - p_{||}}{B^2/4\pi}\right). \tag{7}$$

Plasma turbulence excited by the anisotropic thermal motions rapidly reduces $p_\perp - p_{||}$ in all but the most rarefied plasmas, so that usually

$$\rho\frac{d\mathbf{u}_\perp}{dt} \cong -\nabla_\perp(p_\perp + \frac{B^2}{8\pi}) + \frac{[(\mathbf{B}\cdot\nabla)\mathbf{B}]_\perp}{4\pi} \tag{8}$$

for slow bulk motion $\mathbf{u}$. At the same time the plasma particles relax toward a uniform distribution along the field, with $\mathbf{E}_{||} << \mathbf{E}_\perp$ and

$$\rho\frac{d\mathbf{u}_{||}}{dt} = -\nabla_{||}p_{||} \tag{9}$$

These equations are the statement of Newton's laws of motion, written in general terms in (4). They are accompanied by the induction equation (5) which follows directly from Faraday's law of induction from the fact that $\mathbf{E}_\perp$ is given as $-\mathbf{u}\times\mathbf{B}/c$ in terms of the electric drift velocity $\mathbf{u} = c\mathbf{E}_\perp \times \mathbf{B}/B^2$ deduced from Newton's equations.

Chew et al.,[1956] provide another important derivation of the macroscopic momentum
equation for a collisionless plasma, with the prescription

$$\frac{d}{dt}\frac{p_\perp}{\rho B} = 0, \quad \frac{d}{dt}\frac{B^5 p_{||}}{p_\perp^3} = 0, \tag{10}$$

for the perpendicular and parallel pressure components based on the transverse and longitudinal invariants of the particle motion.

Consider, then, the assertion in section 3 that the equation of motion (7) can be deduced from Maxwell's equation (1) using the guiding center approximation for the motion of the individual particles to represent the electric current. We start with the well-known result *Watson* [956] that the mean motion $\mathbf{v}$ of the guiding center is

$$\mathbf{v} = \mathbf{u} + (\frac{1}{2}Mw_\perp^2 c/qB^4)\mathbf{B}\times\nabla B^2/2 \tag{11}$$
$$+(Mw_{||}^2 c/qB^4)\mathbf{B}\times[(\mathbf{B}\cdot\nabla)\mathbf{B}]$$

for a particle of mass M, charge q, and thermal velocity components $w_{||}$ and $w_\perp$ parallel and perpendicular to the magnetic field, respectively. The motion parallel to the field is described by

$$(dv/dt)_{||} = -(\frac{1}{2}{w_\perp}^2/B^4)\mathbf{B}\{\mathbf{B}\cdot[(\mathbf{B}\cdot\nabla)\mathbf{B}]\} \tag{12}$$

For a singly ionized gas consisting of N ions and N electrons per unit volume, the current density is the difference of the sum of

$e\mathbf{v}_{\text{ion}}$ and $e\mathbf{v}_{\text{electron}}$ over all the particles in a unit volume, so that with equation (11) for the total motion of each particle, the result is

$$\mathbf{j}_\perp = (c/B^2)\mathbf{B} \times[\nabla p_\perp + [(p_\parallel - p_\perp)/B^2](\mathbf{B}\cdot\nabla)\mathbf{B} + \rho d\mathbf{u}/dt] \qquad (13)$$

when all of the geometrical factors are taken into account. The sum of the factors $\frac{1}{2}Mw_\perp{}^2$ and $Mw_\parallel{}^2$ over all particles in a unit volume provides the perpendicular pressure $p_\perp$ and parallel pressure $p_\parallel$, respectively, irrespective of the thermal velocity distribution. Substituting this expression for $\mathbf{j}_\perp$ into Maxwell's equation (1) yields

$$\frac{\partial \mathbf{E}_\perp}{\partial t} = -(4\pi c/B^2)\mathbf{B}\times\{\rho d\mathbf{u}/dt + \nabla(p_\perp + B^2/8\pi) - [1/4\pi + (p_\perp - p_\parallel)/B^2][(\mathbf{B}\cdot\nabla)\mathbf{B}]\}. \qquad (14)$$

Given that $\mathbf{u} = c\mathbf{E}_\perp \times \mathbf{B}/B^2$ it follows that the left hand side of this equations is small compared to the term $\rho d\mathbf{u}/dt$ on the right hand side by the factor $B^2/8\pi\rho c^2$ or u^2/c^2, so it may be set equal to zero. The result is the momentum equation (3.4) and (4.2) again [*Parker*, 1957].

The essential point of these calculations is that the motion of the individual electrons and ions is automatically such as to satisfy Ampere's law relating $\mathbf{j}$ and $\nabla\times\mathbf{B}$ (neglecting terms of the order of u^2/c^2 or $B^2/8\pi\rho c^2$ compared to one) if we accept the fact that the bulk motion satisfies the usual momentum equation.

There are special conditions, of course, for example the energetic particles in the outer magnetosphere and in the geotail, where the guiding center approximation is not useful. In such circumstances a complete formal treatment of the relation between Newton's and Maxwell's equations falls back on the entirely general considerations of Poynting's theorem, taken up in the next section.

5. Electromagnetic Force, Momentum, and Energy

The intimate relation between Newton's and Maxwell's equations described in section 3 and section 4 is implied by Poynting's classical theorem, establishing the equivalence of mechanical momentum and energy and electromagnetic momentum and energy. We provide a brief review of the basic principles. Newton's equation for electrically charged particles in an electromagnetic field $\mathbf{E}$, $\mathbf{B}$ can be written

$$\rho\frac{d\mathbf{v}}{dt} = \delta(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}) \qquad (15)$$

where $\rho(\mathbf{r},t)$ is the mass density and $\delta(\mathbf{r},t)$ is the charge density. The individual elementary particles are represented by moving localized maxima in $\rho(\mathbf{r},t)$ and moving localized extrema in $\delta(\mathbf{r},t)$ coinciding with the maxima in ρ. Thus $\rho(\mathbf{r},t)$ and $\delta(\mathbf{r},t)$ are bounded continuous functions of space and time, with characteristic scales of 10^{-13}cm at the location of the individual electrons and ions, and otherwise zero. The velocity $\mathbf{v}(\mathbf{r},t)$ is defined only at the location of the individual particles, with its value in the interstices of no physical interest. We start with Maxwell's equations

$$\nabla\cdot\mathbf{E} = 4\pi\delta, \nabla\cdot\mathbf{B} = 0, \qquad (16)$$

$$\frac{\partial\mathbf{B}}{\partial t} = -c\nabla\times\mathbf{E},\ 4\pi\mathbf{j} + \frac{\partial\mathbf{E}}{\partial t} = +c\nabla\times\mathbf{B}. \qquad (17)$$

Note, then, that $\mathbf{j} = \mathbf{v}\delta$ on the right hand side of (15).

Use (16) to eliminate δ and (17) to eliminate $\mathbf{j}$. Add the term $\mathbf{B}\nabla\cdot\mathbf{B}/4\pi$ to the right-hand side of (15) to preserve symmetry, with the final result

$$\rho\frac{d\mathbf{v}}{dt} = \frac{\mathbf{E}\nabla\cdot\mathbf{E} + \mathbf{B}\nabla\cdot\mathbf{B} + (\nabla\times\mathbf{E})\times\mathbf{E} + (\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi} - \frac{\partial}{\partial t}\frac{\mathbf{E}\times\mathbf{B}}{4\pi c}, \qquad (18)$$

which can be rewritten as

$$\rho\frac{dv_i}{dt} + \frac{\partial}{\partial t}\frac{P_i}{c^2} = \frac{\partial M_{ij}}{\partial x_j} \qquad (19)$$

where the Maxwell stress tensor M_{ij} is

$$M_{ij} = -\delta_{ij}\frac{E^2 + B^2}{8\pi} + \frac{E_iE_j + B_iB_j}{4\pi} \qquad (20)$$

and P_i is the Poynting vector

$$P_i = c\epsilon_{ijk}E_jB_k/4\pi, \qquad (21)$$

where in ϵ_{ijk} is the usual permutation tensor. It is convenient to rewrite the left-hand side of (18) in terms of the particle momentum flux ρv_i and the Reynolds stress tensor R_{ij} given by (3) so that

$$\frac{\partial}{\partial t}\left(\rho v_i + \frac{P_i}{c^2}\right) = \frac{\partial M_{ij}}{\partial x_j} + \frac{\partial R_{ij}}{\partial x_j}. \qquad (22)$$

It is evident by inspection that P_i/c^2 represents the momentum density of the electromagnetic field, while $M_{ij} + R_{ij}$ represents the total stress or momentum flow field. It must be appreciated that (22) applies to every point in space, where ρ may or may not vanish.

The electromagnetic energy density and energy flux follow similarly beginning with the Newtonian energy statement

$$\begin{aligned}\frac{\partial}{\partial t}(\frac{1}{2}\rho v^2) &= \delta\mathbf{v}\cdot\mathbf{E}\\ &= \mathbf{j}\cdot\mathbf{E}\\ &= \left(\frac{c}{4\pi}\nabla\times\mathbf{B} - \frac{1}{4\pi}\frac{\partial\mathbf{E}}{\partial t}\right)\cdot\mathbf{E}\\ &= \frac{c}{4\pi}[\mathbf{B}\cdot\nabla\times\mathbf{E} - \nabla\cdot(\mathbf{E}\times\mathbf{B})] - \frac{\partial}{\partial t}\frac{E}{8\pi}.\end{aligned}$$

Since $c\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$, this can be rewritten as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \frac{E^2 + B^2}{8\pi}\right) + \nabla\cdot\mathbf{P} = 0, \qquad (23)$$

where again $\mathbf{P}$ is the Poynting vector $c\mathbf{E}\times\mathbf{B}/4\pi$, given by (21). It is evident by inspection that the electromagnetic energy density is $(E^2 + B^2)/8\pi$ and the electromagnetic energy flux is represented by the Poynting vector. Thus Poynting established the mechanical equivalence of electromagnetic and particle stress and momentum fields, with the consequences enumerated in section 4.

Now when $\mathbf{E} = -\mathbf{u}\times\mathbf{B}/c$, it is clear that the electrical contribution to the momentum and energy is small to second order in v/c compared to the magnetic contribution. Neglecting second-order terms, (20) reduces to (2) which states that the magnetic field possesses an isotropic pressure $B^2/8\pi$ and a tension $B^2/4\pi$ in the single direction along the magnetic field. It is easy to show formally from the virial equations that the net effect, averaged over all three dimensions, is dominated by the isotropic pressure [*Parker*, 1953, 1954, 1969b, 1979]. That is to say, a magnetic field, if left to itself, expands to fill all of the available space, thereby minimizing its energy. The solid Earth provides the forces that anchor the geomagnetic field, of course, and the quiet-day field in the region around Earth adjusts itself into the minimum energy configuration consistent with that constraint and the confining magnetopause.

Finally, note that the foregoing results apply at every point in the electromagnetic field, and therefore they apply to any averaging over many particles to provide a macroscopic fluid formulation. The averaging over the nonlinear terms, for example B_iB_j, etc. omits the second order terms $\delta B_i\delta B_j$ where δB_i, represents the local deviation from the mean caused by the individual particle. The standard derivation of Poynting's theorem in most textbooks [cf. *Panofsky and Phillips*, 1955; *Jackson*, 1975] is formulated in terms of the integral of the momentum and energy over an arbitrary fixed volume V. The matter is treated as continuous fluid and $\rho, \delta, \mathbf{v}, \mathbf{j}, \mathbf{E}$, and $\mathbf{B}$ are treated as smooth continuous functions of space and time. Then, since the shape, size, and location of V is arbitrary, it is pointed out that the integrand, namely equations (19) and (23), must be satisfied at every point in space.

In the circumstance that $\mathbf{E} = -\mathbf{u}\times\mathbf{B}/c$, it follows that the Poynting vector reduces to

$$\begin{aligned}\mathbf{P} &= \mathbf{B}\times(\mathbf{u}\times\mathbf{B})/4\pi\\ &= \mathbf{u}_\perp B^2/4\pi.\end{aligned} \tag{24}$$

The Poynting vector represents the transport of magnetic enthalpy $B^2/4\pi$ with the electric drift velocity $\mathbf{u}$ in the direction perpendicular to the magnetic field. It illustrates again the bodily transport of the field in the moving fluid.

It is instructive to derive this same result directly from the mechanical work done by the Maxwell stress carried in the moving fluid. The work done on an element of area dS_j by the motion u_i in oppositon to the Maxwell stress is

$$-u_iM_{ij}dS_j = \left(u_j\frac{B^2}{8\pi} - \frac{u_kB_kB_j}{4\pi}\right)dS_j$$

where the Maxwell stress tensor is given by (2), or by (20) upon noting that the terms in E^2 are small to second order in u/c compared to the terms in B^2. The enthalpy flux is the sum of the convective transport of magnetic energy $u_jB^2/8\pi$ and the rate at which work is done by u_j, so that the energy or enthalpy flow is $(u_jB^2 - u_kB_kB_j)dS_j/4\pi$. To obtain the energy flow across dS_j from (24), note that the velocity parallel to B_i can be written $u_kB_kB_i/B^2$. It follows that $u_\perp$ can be written $u_i - u_kB_kB_i/B^2$, so that (24) gives the same result. Thus the transport of energy is in the direction perpendicular to the magnetic field and includes the work done by the Maxwell stress in addition to the convective transport of magnetic energy.

Application to a plane transverse Alfven wave in an inviscid infinitely conducting incompressible fluid of uniform density ρ shows how the energy is transported in that simple case. Consider a plane Alfven wave with velocity $u(z,t)$ and magnetic field $b(z,t)$ in the y direction propagating along the uniform field B_0 in the z direction. It is easy to show that the exact dynamical equations are

$$\rho\frac{\partial u}{\partial t} = \frac{B_0}{4\pi}\frac{\partial b}{\partial z},\quad \frac{\partial b}{\partial t} = B_0\frac{\partial u}{\partial z}. \tag{25}$$

Then for the wave

$$u(z,t) = U sin[\omega(t - z/C) + \varphi], \tag{26}$$

where $C = B_0/(4\pi\rho)^{\frac{1}{2}}$ and φ is an arbitrary constant, it follows that

$$b(z,t) = -u(z,t)(4\pi\rho)^{\frac{1}{2}}. \tag{27}$$

The magnetic field has y and z components b and B_0, respectively, so that the magnitude of the field is

$$B(z,t) = [B_0^2 + 4\pi\rho U^2(z,t)]^{\frac{1}{2}}. \tag{28}$$

The energy transported by the wave can be computed in three obvious ways. First of all, the group velocity is the same as the phase velocity C, so the energy flux I can be written

$$I = [\frac{1}{2}\rho < v^2 > + < b^2 > /8\pi]C,$$

representing the energy density of the wave propagating at the Alfven speed, where angular brackets indicate the time average. With (26) and 27) it follows that

$$\begin{aligned}I &= \rho < u^2 > C,\\ &= \frac{1}{2}\rho U^2 C.\end{aligned} \tag{29}$$

On the other hand, the rate at which the Maxwell stress does work across any surface $z = const$ is

$$I = - < ub > B/4\pi, \tag{30}$$

which comes out the same, of course.

Finally, the energy transport can be computed directly from (24) noting that

$$\begin{aligned}\mathbf{u}_\perp &= \mathbf{u} - \mathbf{u}\cdot\mathbf{B}/B\\ &= \mathbf{e}_y u(1 - b^2/B^2) - \mathbf{e}_z ub/B\\ &= \mathbf{e}_y uB_0^2/B^2 - \mathbf{e}_z ub/B.\end{aligned} \tag{31}$$

Thus

$$\mathbf{u}_\perp B^2/4\pi = \mathbf{e}_y uB_0^2/4\pi - \mathbf{e}_z ubB/4\pi. \tag{32}$$

The time average of the y component is zero and the z component gives the result already obtained. Thus (24) tells us that the magnetic energy $B_0^2/4\pi$ is carried in the tranverse oscillation in the y direction, but represents no net transport. There is a net transport in the z direction because $\mathbf{u}_\perp$ has a nonnegative forward pointing z component (even though $\mathbf{u}$ does not), given by

$$-\frac{ub}{B} = \frac{U^2}{C}sin^2[\omega(t - z/C) + \varphi]. \tag{33}$$

So the energy follows a serpentine or sinusoidal path but always progresses in the direction of propagation of the wave.

6. Application to Stationary States

The applications of the $\mathbf{E},\ \mathbf{j}$ and $\mathbf{B},\ \mathbf{v}$ paradigms to simple cases of stationary currents and magnetic fields show the relative efficiency of the $\mathbf{E},\ \mathbf{j}$ paradigm in treating certain symmetric cases, as well as the limitations and errors arising in nonsymmetric configurations. In fact, the efficiency of the $\mathbf{E},\ \mathbf{j}$ paradigm arises precisely because it avoids the basic dynamics, concentrating on the simple conservation laws applied to the geometry of a symmetric unperturbed magnetic field. As we shall see, the scheme fails in the absence of symmetry when we come to compute the perturbed field configuration, because the field aligned currents are necessarily aligned along the unperturbed field. The $\mathbf{B},\ \mathbf{v}$ paradigm avoids the difficulty because it computes the field perturbations from stress balance, finally deducing the currents from the perturbed field with the aid of Ampere's law. We shall see that the topology of the currents is different from results obtained with the $\mathbf{E},\ \mathbf{j}$ paradigm.

The applications begin in subsection 6.1 with the well known axially symmetric problem of a circular band of trapped ions, that is, a uniform ring current, circling Earth, to illustrate the methods of the two paradigms, showing the efficiency of the $\mathbf{E},\ \mathbf{j}$ approach relative to the $\mathbf{B},\ \mathbf{v}$ paradigm in such simple circumstances. Then subsection 6.2 goes on to the nonsymmetric problem of a localized equatorial cluster of trapped ions, inflating a single small flux bundle in a dipole field. We employ both paradigms again, obtaining results that are quite different when it comes to the path of the perturbed field lines and the associated currents. In view of the importance of this result the presentation treats the problem in some detail.

Finally, section 6.3 considers the problem of an elemental flux bundle inflated with an isotropic particle distribution within a dipole field. The displacement of the flux bundle is computed to show the use of the optical analogy in the $\mathbf{B},\ \mathbf{v}$ paradigm. The $\mathbf{E},\ \mathbf{j}$ paradigm is not applicable for the reason illustrated in sub-section 6.2.

6.1. Equatorial Band of Collisionless Particles

Consider a uniform thin equatorial band of energetic ions circling at a radial distance $\varpi = (x^2+y^2)^{\frac{1}{2}} = a$ around a three-dimensional magnetic dipole located at the origin and pointing in the negative z direction so that the magnetic field $B(\varpi)$ at the position of the ions is in the positive z direction. The mass of each ion is M and the charge is e. The ions all have the same velocity w perpendicular to the magnetic field (pitch angle $\frac{1}{2}\pi$). There are n ions per unit length around the band, for a total of $\mathcal{N} = 2\pi a n$. An equal number of cold electrons is present to preserve charge neutrality. The total kinetic energy of the ions is

$$\mathcal{E} = \mathcal{N}\frac{1}{2}Mw^2$$

$$= \mathcal{N}\mu B(a)$$

where μ is the diamagnetic moment $\frac{1}{2}Mw^2/B(a)$ of the individual ion and $B(a)$ is the magnetic field at the radial distance $\varpi = a$ in the equatorial plane. Elsewhere in the equatorial plane $B(\varpi) = B(a)(a/\varpi)^3$. Assume that $\mathcal{E}$ is sufficiently small that the magnetic perturbation $\Delta\mathbf{B}(\mathbf{r})$ is everywhere small compared to the unperturbed field $\mathbf{B}\ (\mathbf{r})$.

The easiest way to calculate the magnetic perturbation is to use the $\mathbf{E},\ \mathbf{j}$ paradigm noting that the total azimuthal current is nev_D where v_D is the ion gradient drift velocity given by (11) as

$$v_D = \frac{c\mu}{eB(a)}\left|\frac{dB}{da}\right| \tag{34}$$

$$= 3c\mu/ea \tag{35}$$

westward. The azimuthal drift current is

$$I_D = 3cn\mu/a, \tag{36}$$

around the ring of radius a, producing the perturbation field in the neighborhood of the origin with z component

$$\Delta B_D = -2\pi I_D/ca$$

$$= -3\mu\mathcal{N}/a^3. \tag{37}$$

There is an additional contribution ΔB_μ from the diamagnetic moment μ of each ion in the amount μ/a^3 per particle, or

$$\Delta B_\mu = +\mu\mathcal{N}/a^3. \tag{38}$$

in the neighborhood of the origin. The diamagnetic effect of the cyclotron motion of the individual ion tends to exclude the magnetic field from the region of the ions, thereby compressing the field slightly elsewhere. So ΔB_μ represents a field in the positive z direction. The total ΔB in the neighborhood of the origin is the algebraic sum of ΔB_D and ΔB_μ yielding the reduction

$$\Delta B_z = -2\mu\mathcal{N}/a^3 \tag{39}$$

in the z component of the dipole field in the neighborhood of the origin.

If ΔB_z is a short-lived perturbation, it does not penetrate deeply into Earth, which acts, then, as a diamagnetic sphere of radius $R(<< a)$. The result is the perturbation magnetic field

$$\Delta B_r(r,\theta) \cong -\frac{2\mu\mathcal{N}}{a^3}\left(1-\frac{R^3}{r^3}\right)cos\theta$$

$$\Delta B_\theta(r,\theta) \cong \frac{2\mu\mathcal{N}}{a^3}\left(1+\frac{R^3}{2r^3}\right)\sin\theta \tag{40}$$

in spherical polar coordinates (r,θ,φ) in the vicinity of Earth ($R < r << a$). The reduction of the horizontal component of the field of the equator ($z=0, \theta=\frac{1}{2}\pi$) at the surface of Earth is $3\mu\mathcal{N}/a^3$.

The efficiency of the $\mathbf{E},\ \mathbf{j}$ paradigm is evident in its direct path to the final result. However, it is not without interest to obtain some idea of the forces that deform the dipole field to achieve the expansion of the field in the vicinity of Earth. As a beginning, note that (39) can be rewritten as

$$\frac{\Delta B_z}{B_o} = \frac{2\mathcal{E}}{3\mathcal{E}_E} \tag{41}$$

where $\mathcal{E}_E = \frac{1}{3}B_D^2R^3$ is the magnetic energy in the external ($r > R$) unperturbed dipole field with intensity B_o at the equator at the surface of Earth, $r = R, \theta = \frac{1}{2}\pi$ [*Dessler* and *Parker*, 1959, 1968]. In the presence of the diamagnetic Earth, the fractional reduction of the horizontal component at the equator at the surface of

Earth is precisely $\mathcal{E}/\mathcal{E}_E$, the fraction of the total magnetic energy represented by the kinetic energy of the ions.

Consider, then, the $\mathbf{B}$, $\mathbf{v}$ paradigm, with its focus on Maxwell stress and the geometric distortion of the magnetic field by the pressure of the energetic ions. The direct approach is through formal solution of the magnetostatic equation, putting $u_i = 0$ in equation (3.4) to obtain

$$\frac{\partial}{\partial x_j}(p_{ij} + M_{ij}) = 0. \tag{42}$$

With axial symmetry this reduces to the quasi-linear Grad-Shafranov equation, whose solution is not elementary because of the unknown, arbitrary, and generally nonlinear function of the vector potential. So, as with the $\mathbf{E}$, $\mathbf{j}$ paradigm, we take advantage of the fact that the system is only slightly perturbed. Then, applying Ampere's law to the Biot-Savart integral, the perturbation $\Delta\mathbf{B}\ (\mathbf{r})$ is

$$\Delta B(\mathbf{r}) = \frac{1}{4\pi}\int \frac{d^3\mathbf{r}'[\nabla\times\mathbf{B}(\mathbf{r}')]\times(\mathbf{r}-\mathbf{r}')}{|\,\mathbf{r}-\mathbf{r}'\,|^3}.$$

The equation for magnetostatic equilibrium in the presence of the force $\mathbf{F}$ per unit volume exerted by the particles and plasma on the field $\mathbf{B}$ is

$$4\pi\mathbf{F} + (\nabla\times\mathbf{B})\times\mathbf{B} = 0.$$

The vector product with $\mathbf{B}$ results in

$$(\nabla\times\mathbf{B})_\perp = 4\pi\mathbf{F}\times\mathbf{B}/B^2$$

where the subscript indicates the component perpendicular to $\mathbf{B}$. Symmetry decrees that there is no torsion in the field $(\nabla\times\mathbf{B})_\parallel = 0$, so the Biot-Savart integral for $\Delta\mathbf{B}(\mathbf{r})$ becomes

$$\Delta\mathbf{B}(\mathbf{r}) =$$

$$\int d^3\mathbf{r}'\frac{\mathbf{B}(\mathbf{r}')[\mathbf{F}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')] - \mathbf{F}(\mathbf{r}')[\mathbf{B}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')]}{B^2(\mathbf{r}')\,|\,\mathbf{r}-\mathbf{r}'\,|^3} \tag{43}$$

Specification of the force $\mathbf{F}\ (\mathbf{r})$ exerted on the fields gives the deformation $\Delta\mathbf{B}\ (\mathbf{r})$ [*Parker*, 1962].

The similarity of the mathematics to the Biot-Savart integral for $\Delta\mathbf{B}\ (\mathbf{r})$ in the $\mathbf{E}$, $\mathbf{j}$ paradigm is obvious, of course. In that paradigm we would write $\mathbf{F} = \mathbf{B}\times\mathbf{j}/c$, from which it follows that $\mathbf{j}_\perp = c\mathbf{F}\times\mathbf{B}/B^2$. Thus, for instance, a ring current I of radius a produces a magnetic field $2\pi I/ca$ at the center of the ring. In terms of the total outward radial force $\mathcal{F} = IB(a)/c$ per unit length around the ring, the field at the origin is

$$\Delta\mathbf{B}(0) = -\frac{2\pi\mathcal{F}}{aB(a)} \tag{44}$$

The force $\mathcal{F}$ is a consequence of the diamagnetic expulsion of the dipole moment μ of each ion, which is opposed by the Lorentz force $IB(a)/c$. The force $\mathcal{F}_i$ is exerted on a magnetic dipole moment μ_i by the field B_j is $\mu_j\partial B_i/\partial x_j$. In the present instance, with μ pointing in the z direction, this is $\mu\partial B_\varpi/\partial z = 3\mu B(a)/a$ per ion, so that

$$\mathcal{F} = 3n\mu B(a)/a.$$

As a consequence of this force, the ion drifts in the azimuthal direction with a velocity v_D such that $qv_D B(a) = c\mathcal{F}$, yielding (35). In other words, the ring current I arises in order that the inward Lorentz force $IB(a)/c$ balance the diamagnetic repulsion.

Now for an equatorial band of ions with pitch angle $\frac{1}{2}\pi$, the pressure tensor has the two nonvanishing components $p_{\varphi\varphi} = p_{\varpi\varpi}$, each of which is equal to the kinetic energy density of the ions. With ϖ representing radial distance $(x^2+y^2)^{\frac{1}{2}}$ from the z axis, it follows that $\mathbf{r}-\mathbf{r}' = -e_\varpi\varpi$ for the origin $(\mathbf{r} = 0)$. Then $\mathbf{B}(\mathbf{r}) = e_z B_o(R/\varpi)^3$ and $\mathbf{B}\cdot(\mathbf{r}-\mathbf{r}') = 0$, so that

$$\Delta\mathbf{B}(0) = e_z\frac{2\pi}{B_E R_E^3}\int_{-\varpi}^{+\varpi}dz\int_{R_E}^{\varpi}d\varpi\varpi^2\frac{\partial p_{\varpi\varpi}}{\partial\varpi}.$$

Integrating by parts, with $p_{\varpi\varpi}$ nonvanishing except in a small neighborhood of $\varpi = a$, yields

$$\Delta\mathbf{B}(0) = -e_z\frac{2\mathcal{E}}{B_o R^3} \tag{45}$$

where

$$\mathcal{E} = 2\pi\int_{-\varpi}^{+\varpi}dz\int_{R_E}^{\varpi}d\varpi\varpi\, p_{\varpi\varpi} \tag{46}$$

represents the total kinetic energy of the ions. This result is just (41) again.

It is instructive to look more closely at the forces exerted by the equatorial band of ions at $\varpi = a$. Denoting the small radial width of the band by $h(<< a)$, the ions exert a total radial force $\mathcal{F} = (n/h)\frac{1}{2}Mw^2$ per unit length inward at the inner edge ($\varpi = a$) of the band and outward at the outer edge ($\varpi = a+h$, assuming that the ions are distributed uniformly across h). It follows from (44) that

$$\begin{aligned}\Delta\mathbf{B}(0) &= 2\pi\mathcal{F}\left[\frac{1}{aB(A)} - \frac{1}{(a+h)B(a+h)}\right]\\ &\cong \frac{4\pi\mathcal{F}h}{a^2B(a)}\left[1+O\left(\frac{h}{a}\right)\right]\\ &= 2\mu\mathcal{N}/a^3\end{aligned} \tag{47}$$

in which $\mu = \frac{1}{2}Mw^2/B(a)$, which is the same result as (39), of course. The point is that the net effect of the same force per unit length exerted in opposite directions at a and $a+h$ arises from the longer circumference and the weaker field at $a+h$.

It is also instructive to examine the forces exerted on the ions in the small angular sector $\Delta\varphi$ of the band. At $\varpi = a$ there is an outward radial force $a\Delta\varphi\mathcal{F}$. At $\varpi = a+h$, the force is inward and of magnitude $(a+h)\Delta\varphi\mathcal{F}$. The compressive particle force $\mathcal{F}h$ is exerted in the azimuthal direction around the radius a of the band, providing a net outward radial force $\mathcal{F}(h/a)a\Delta\varphi$. Thus the total radial force is zero, so that the ions in the band are in quasi-static equilibrium with the magnetic field $\mathbf{B}(\mathbf{r})$, with the Lorentz force of the drift current nqv_D/c balancing the diamagnetic repulsion.

6.2. An Equatorial Clump of Particles

Consider a small cluster of $\mathcal{N}$ energetic ions with mass M, charge e, and velocity w with pitch angle $\frac{1}{2}\pi$ and individual diamagnetic moment μ at a radial distance a in the equatorial plane of a two dimensional magnetic dipole field. An equal number of cold electrons is assumed as well as a tenuous background of collisionless thermal plasma. The dipole, with moment $B_o R^2$ per unit length in the z direction, extends uniformly along the x axis $(y = z = 0)$ and points in the positive z direction, producing the magnetic field

$$B_\varpi = +B_o\left(\frac{R}{\varpi}\right)^2 sin\varphi, B_\varphi = -B_o\left(\frac{R}{\varpi}\right)^2 cos\varphi \quad (48)$$

in cylindrical polar coordinates $\varpi = (y^2+z^2)^{\frac{1}{2}}$ and φ (measured around the x axis from the y axis). In Cartesian coordinates,

$$B_y = \frac{2B_oR^2yz}{(y^2+z^2)^2},\ B_z = \frac{B_oR^2(z^2-y^2)}{(y^2+z^2)^2}. \quad (49)$$

It is convenient to think of B_o as the polar field at the surface of a two-dimensional Earth, $\varpi = R$. The magnitude of the field is

$$B = B_o(R/\varpi)^2 \quad (50)$$

and the field lines are the circles

$$\varpi = 2\lambda\ cos\varphi \quad (51)$$

with radius λ and the centers on the y axis at $y = \lambda$ so that they are tangent to the z axis at $y = z = 0$.

Suppose, then, that the $\mathcal{N}$ ions are spread uniformly over a small rectangle in the equatorial plane $(z = 0)$ at $x = 0,\ y = a$. The rectangle has side h in the radial y direction and side l in the x direction $(h, l << a)$ with a surface number density ν so that $\mathcal{N} = \nu lh$. The magnetic field at the position of the particles is $B(a) = B_oR^2/a^2$ and points in the negative z direction.

The $\mathbf{E}, \mathbf{j}$ paradigm begins by noting [*Stern*, 1992] that the ions have the gradient drift

$$v_D = \frac{c\mu \mid B'(a) \mid}{eB(a)} \quad (52)$$

$$v_D = +\frac{2c\mu}{ea} \quad (53)$$

in the negative x direction. The surface current density within the rectangle of ions is

$$J = \nu e v_D \quad (54)$$

$$J = \frac{2cU}{aB(a)} \quad (55)$$

in the negative x direction where U is the particle kinetic energy per unit area,

$$U = \frac{1}{2}\nu M w^2$$

$$= \nu\mu B(a). \quad (56)$$

The total particle energy $\mathcal{E}$ is then hlU.

Assuming that there are enough background thermal electrons, the electrostatic forces maintain local charge neutrality. If follows from conservation of charge that the drift current J bifurcates into two field-aligned currents, each of surface density $J_\parallel = \frac{1}{2}J$ flowing away along the fieldlines from the back end (toward negative x) of the rectangle. It also follows that two field aligned currents $J_\parallel$ converge from each side to flow into the front end of the rectangle, sketched in Figure 1. The closure of the current system across the ionosphere in the neighborhood of the dipole at the origin is assumed to be nondissipative, so that U is conserved for a steady state. (In fact, the terrestrial ionosphere is dissipative and the energy $\mathcal{E}$ is slowly degraded [*Parker*, 1966]). In the present idealized (dissipationless) state, it follows that

$$J_\parallel = cU/aB(a). \quad (57)$$

Stern [1992] suggests that the region 1 Birkeland currents may be precisely this $J_\parallel$ produced by energetic particles trapped in the geomagnetic tail.

The next question is the nature of the geomagnetic perturbation produced by $J_\perp$ and $J_\parallel$. The current $J_\parallel$ flows along the field lines on the surface $x = 0$, l between the two circles

$$\varpi = acos\varphi, \varpi = (a+h)cos\varphi. \quad (58)$$

With $J_\parallel$ flowing along the field, it follows (see Figure 1) that the ribbon of current $J_\parallel$ has a width $h(\varpi)$ such that $h(\varpi)B(\varphi) = const$, while conservation of current requires $h(\varpi)J_\parallel(\varpi) = const$. Thus

$$J_\parallel(\varpi) = J_\parallel(a)(a/\varpi)^2 \quad (59)$$

For $h, l << a$ the field around $J_\parallel$ is the field produced by two antiparallel ribbons of electric current of width $h(\varpi)$ separated by a fixed distance l, with $dh/d\varpi = 2h/\varpi << 1$.

The magnetic field of two antiparallel ribbons of surface current density $J_\parallel$ with width 2b and separation 2d is readily expressed in terms of the local Cartesian coordinates (ξ, η, ζ), sketched in Figure 1, where ζ represents distance parallel to the ribbons, η is distance measured from the midplane between the two and ξ is distance measured across the width 2b of the ribbon from the line up the middle of each ribbon. The result is the scalar magnetic potential

$$\phi = \frac{J_\parallel}{c}\left\{(b-\xi)\ell n\frac{(b-\xi)^2+(\eta-d)^2}{(b-\xi)^2+(\eta+d)^2}+\right.$$
$$(b+\xi)\ell n\frac{(b+\xi)^2+(\eta-d)^2}{(b+\xi)^2+(\eta+d)^2}$$
$$+2(\eta-d)\left[tan^{-1}\frac{b-\xi}{\eta-d}+tan^{-1}\frac{b+\xi}{\eta-d}\right]$$
$$\left.-2(\eta+d)\left[tan^{-1}\frac{b-\xi}{\eta+d}+tan^{-1}\frac{b+\xi}{\eta+d}\right]\right\}$$

describing the field at position (ξ, η) for $\xi^2, \eta^2 << a^2$. At greater distance from the ribbon the curvature and the slow variation of the width h with ζ (along the field lines) must be taken into account. The two ribbons of $J_\parallel$ lie locally in the planes $\eta = \pm d$ $(\ell = 2d)$ and the edges of the ribbons are at $\xi = \pm b (h(\varpi) = 2b)$. The field component B_ξ through the space between the ribbons is $-\partial\phi/\partial\xi$,

$$B_\xi = \frac{J_\parallel}{c}\times$$

$$\left[tan^{-1}\frac{b-\xi}{d-\eta}+tan^{-1}\frac{b+\xi}{d-\eta}+tan^{-1}\frac{b-\xi}{d+\eta}+tan^{-1}\frac{b+\xi}{d-\eta}\right]. \quad (60)$$

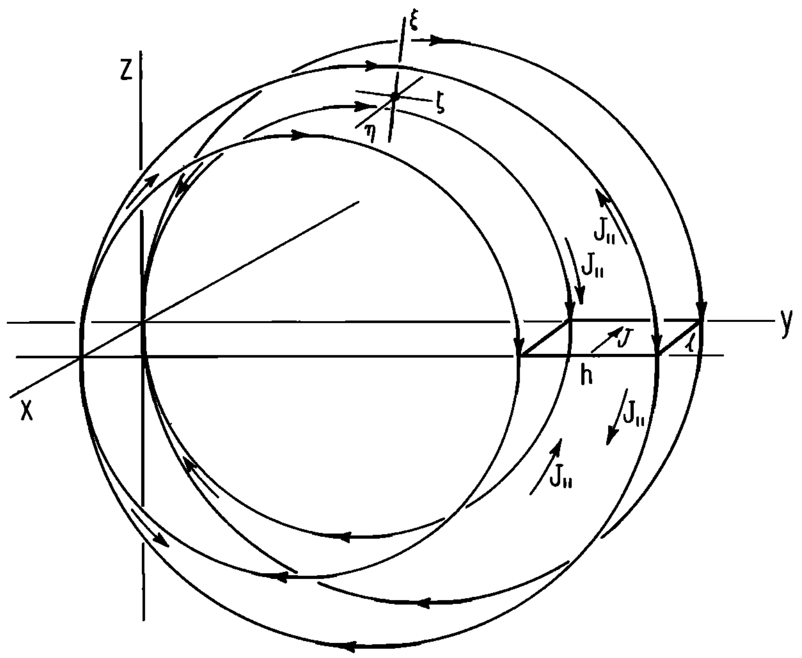

Figure 1. A schematic drawing of the unperturbed field lines $\varpi = a\,cos\varphi$ and $\varpi = (a+h)\,cos\varphi$ at $x = 0, l$ for the two-dimensional dipole at the origin. The rectangle $h \times l$ of ions at $a < y < a + h, z = 0$ provides the surface current J, which divides into the field-aligned currents $J_{\parallel}$ flowing away in both directions from the back and into the front ends of the cluster of ions. The indicated local coordinate system (ξ, η, ζ) is used in (60) and (61) for treating the magnetic fields produced by $J_{\parallel}$.

On the midplane ($\eta = 0$) between the current sheets

$$B_\xi = \frac{2J_{\parallel}}{c}\left(tan^{-1}\frac{b-\xi}{d} + tan^{-1}\frac{b+\xi}{d}\right). \tag{61}$$

with $B_\eta = 0$ from symmetry considerations. In the simple case that $\ell << h$ (i.e. $d << b$), we have $B_\eta = 0$ and

$$B_\xi(\varpi) = 4\pi J_{\parallel}(\varpi)/c \tag{62}$$

throughout the interior of the thin region between the two parallel ribbons of $\pm J_{\parallel}$. Close to the edges of the ribbon the field declines, of course. The field is negligible in the exterior space, except near the edges, and declines as $B_\xi(\xi) \sim (4J_{\parallel}/c)bd/\xi^2$ for $\xi^2 >> b^2$. It follows that the magnetic field extending lengthwise along the ribbons of $J_{\parallel}(\varpi)$ is deflected by the small angle $\vartheta(\varpi)$, where

$$\begin{aligned}\vartheta(\varpi) &= \frac{B_\xi(\varpi)}{B(\varpi)}\\ &= \frac{4\pi J_{\parallel}(\varpi)}{cB(\varpi)}\\ &= \frac{4\pi J_{\parallel}(a)}{cB(a)}\\ &= \frac{4\pi U}{aB(a)^2}\end{aligned} \tag{63}$$

at the radial distance ϖ along the unperturbed field line $\varpi = acos\varphi$. The deflection is constant along the field line because $J(\varpi) \sim B(\varpi)$, so that $\vartheta(\varpi) = \vartheta(a) \equiv \vartheta$. Note, then, that the field is deflected abruptly by the surface current density J in the amount 2ϑ where it passes through the rectangle of energetic ions in the equatorial plane.

The next question is the path of the perturbed field lines, particularly those on which the cluster of equatorial particles is trapped. It is here that $\mathbf{E}$, $\mathbf{j}$ paradigm, that has served so well up to this point, runs into difficulty. The field lines in the midplane $\eta = 0$ in the interior of the region are represented by the family of solutions of the differential equation

$$\begin{aligned}\frac{d\xi}{d\zeta} &= \frac{B_\xi}{B}\\ &= \frac{4\pi J_{\parallel}}{cB}\\ &= \vartheta\end{aligned}$$

for $b >> d(h >> \ell)$. It follows that in a distance of the order of b/ϑ, the field line leaves the region between the ribbons, that is, $\mid \xi \mid > b$. With sufficiently small $h (= 2b)$ this distance is small compared to the length $O(a)$ of the field line. For $\xi >> b$,

$$\frac{d\xi}{d\zeta} \sim \vartheta\left(\frac{b}{\xi}\right)^2$$

in order of magnitude. The deflection falls rapidly below ϑ in $\xi >> b$. The total transverse ξ/b displacement increases asymptotically as $(3\vartheta\zeta/b)^{\frac{1}{3}}$. The perturbed field lines have the form sketched in Figure 2 where they pass through between the two ribbons of current. This result is incorrect, (1) because the perturbed field lines fail to intersect the equatorial cluster of particles that are supposed

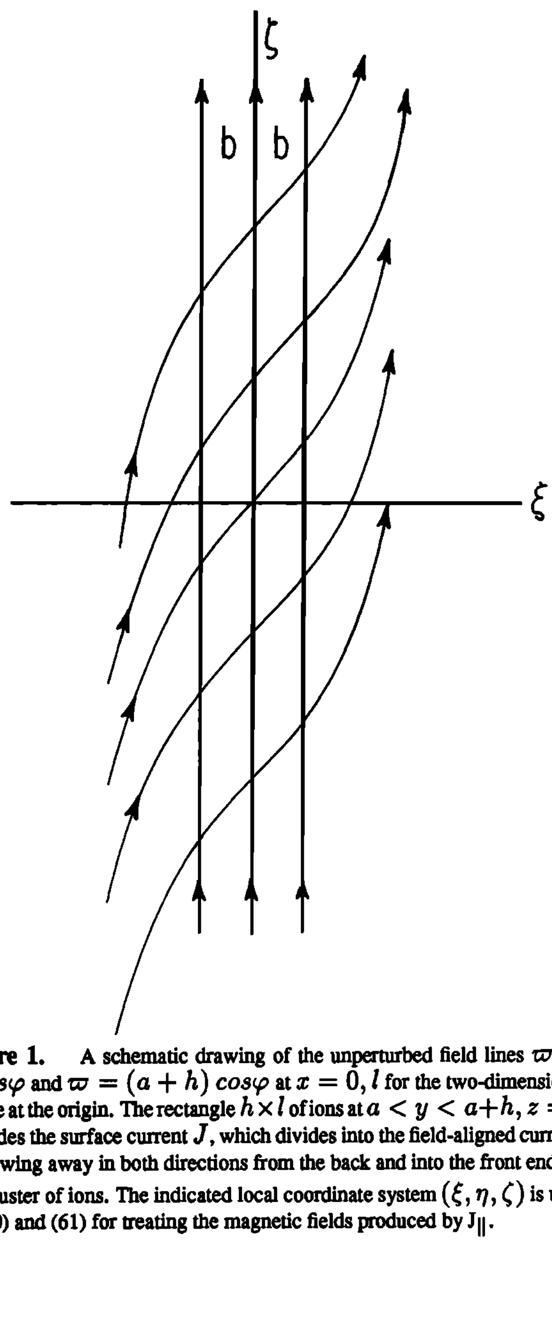

Figure 1. A schematic drawing of the unperturbed field lines $\varpi = a\ cos\varphi$ and $\varpi = (a+h)\ cos\varphi$ at $x = 0, l$ for the two-dimensional dipole at the origin. The rectangle $h \times l$ of ions at $a < y < a+h, z = 0$ provides the surface current J, which divides into the field-aligned currents $J_{||}$ flowing away in both directions from the back and into the front ends of the cluster of ions. The indicated local coordinate system (ξ, η, ζ) is used in (60) and (61) for treating the magnetic fields produced by $J_{||}$.

to be the cause of their deflection, and (2) because field lines in static equilibrium are deflected in the manner shown only upon crossing a region of enhanced pressure, that is, enhanced B. So $J_{||}$ must be tied to the perturbed field, rather than the unperturbed field. That is to say, in the correct solution the current $J_{||}$ partially follows the perturbed field lines, providing the uniform deflection given by (63) everywhere along the perturbed field. The error is qualitative because of the arbitrarily small width of the perturbed flux bundle.

To compute the consequences to first order in the perturbation note that arc length ds along the circle $\varpi = acos\varphi$ is given by $ds = ad\varphi$. The separation $H(\varphi)$ of the perturbed field line from the unperturbed satisfies the condition $dH = \vartheta ds$. If the field line is fixed in the rigid conducting Earth at $\varpi = R = acos\varphi_R, \varphi = \varphi_R$, it follows that $H(\varphi_R) = 0$ and for $0 < \varphi < \varphi_R$,

$$H(\varphi) = a\vartheta(\varphi_R - \varphi). \tag{64}$$

The outward displacement of the ions at the equator ($\varphi = 0$) is, therefore,

$$H(0) = a\vartheta\varphi_R, \tag{65}$$

$$= 4\pi U\varphi_R/B(a)^2,$$

$$= \frac{4\pi U\varphi_R a^4}{B_R{}^2\, R^4}, \tag{66}$$

Note that the width of each ribbon of $J_{||}$ declines as ϖ^2 or sin^2 $(\frac{1}{2}\pi - \varphi)$ as φ increases toward $\pi/2$, so that the idealization that the width $h(\varphi)$ of the ribbons is large compared to their fixed separation ℓ requires that φ_R not be so near to $\frac{1}{2}\pi$ as to violate $cos^2\ \varphi_R >> h(0)/\ell$. It is evident that this calculation is not self-consistent, because the angular deflection ϑ indicates that the field lines are displaced a total distance of the order of $a\vartheta$ from their unperturbed positions, carrying them into regions where the field differs by $\Delta B \sim \vartheta B$. However, this error is as large as the perturbation described by (62), so we cannot expect the result to be quantitatively correct.

So consider the $\mathbf{B}$, $\mathbf{v}$ paradigm, which avoids these problems by computing the perturbed field directly from the equations of stress balance, using the optical analogy. The electric currents then follow from Ampere's law, and we shall find them to be substantially different from those computed using the $\mathbf{E}$, $\mathbf{j}$ paradigm and sketched in Figure 1.

To pursue the $\mathbf{B}$, $\mathbf{v}$ paradigm, we begin by noting that the effect of the trapped particles is a slight inflation of the elemental flux bundle at the equatorial plane. Hence the elemental flux bundle is expelled more strongly at the equator by the pressure gradient in the surrounding magnetic field, that is, by the diamagnetic repulsion. The normal equilibrium balance between the outward force $-\nabla B^2/8\pi$ on each elemental flux bundle and the inward tension force $(\mathbf{B} \cdot \nabla)\mathbf{B}/4\pi = TK$ is upset, where T is the magnetic tension $B^2/4\pi$ and K is the curvature of the field lines. The inflated flux bundle is displaced outward along its entire length so that the sharp curvature across the equatorial apex of the bundle (where the particles are trapped) affords an equilibrium balance of tension against expulsion, sketched in Figure 3. The easiest way to compute the force with which the ambient pressure gradient tends to expel the ions is to note that the diamagnetic moment per unit area $\nu\mu$ is repelled by the ambient dipole field by a force $\mathcal{F}$ per unit area given by

$$\mathcal{F} = \nu\mu\frac{\partial B_y}{\partial z}$$

$$= 2U/a. \tag{67}$$

This force of expulsion is opposed by the magnetic tension in the displaced flux bundle on each side of the cluster of ions in the equatorial plane. The field is deflected by an angle ϑ from the z direction on each side, sketched in Figure 4. So the tension $B(a)^2/4\pi$ provides a total inward force $\vartheta B(a)^2/2\pi$ from both sides of the equatorial plane. Equating this to $\mathcal{F}$ yields (63) at $\varpi = a$. On this point the two paradigms agree.

Now consider the path of the flux bundle from its displaced apex at the equatorial plane inward to the dipole at the origin. In the $\mathbf{B}, \mathbf{v}$ paradigm the path is calculated from the optical analogy [*Parker*, 1981a, b, 1989, 1991, 1994]. The method is not restricted in any way to small deflections and displacements, although we shall carry through the computation in the limit of small deflection to compare directly with the results of the $\mathbf{E}$, $\mathbf{j}$ paradigm. So the $\mathbf{B},\mathbf{v}$ paradigm, employing the exact method of the optical analogy,

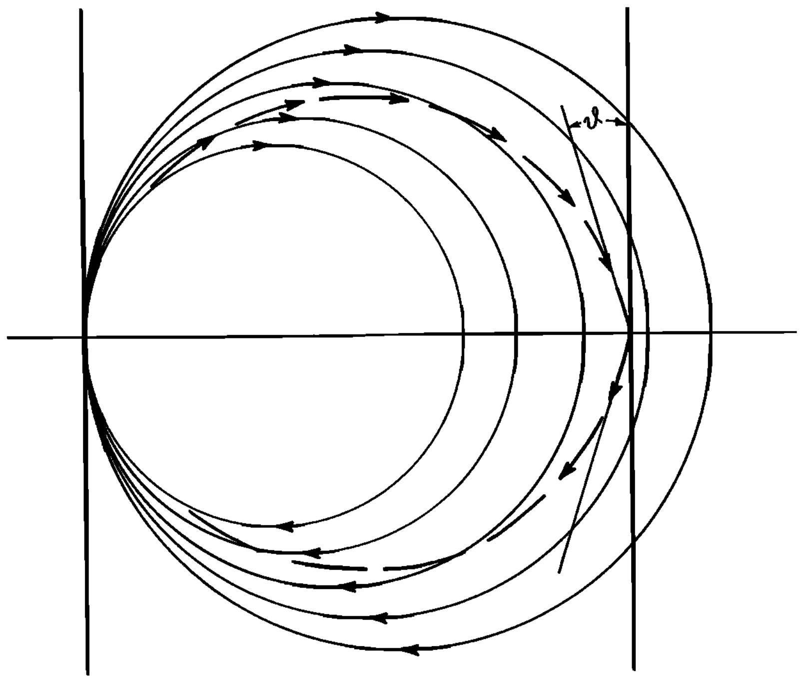

Figure 3. A sketch of the ambient field (solid lines) as a backdrop for the flux bundle (arrows) displaced by the presence of energetic ions trapped at the equatorial plane.

avoids questions of $\mathbf{J}_{\|}$ along perturbed or unperturbed field lines, with $\mathbf{J}$ computed only after the field is fully determined.

The essential point is that the path of any elemental flux bundle, however it may be displaced in the ambient field, is the same as an optical ray path (geodesic) in an index of refraction proportional to the magnitude $B \sim (a/\varpi)^2$ of the ambient field. In the present case, pressure balance guarantees that B within the elemental flux bundle is the same as the ambient field outside, except at the location of the ions in the equatorial plane. So the path of the bundle is described by Fermat's principle

$$\delta \int dsB = 0.$$

Euler's equation reduces to

$$K = \partial lnB/\partial s_{\perp}, \tag{68}$$

where K is curvature of the field and $s_{\perp}$ denotes distance per-

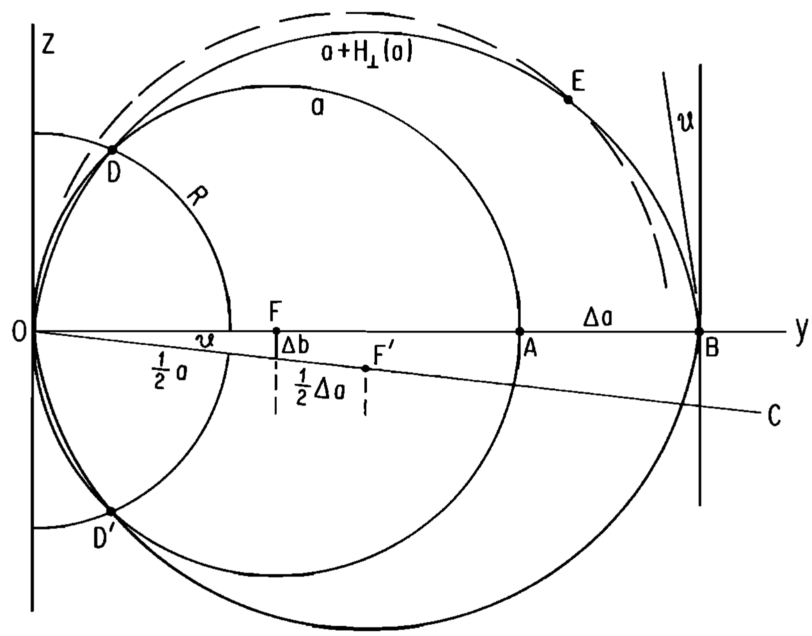

Figure 4. A drawing of the displace flux bundle ADEB (given by (71)) against the ambient field line (dashed curve) through the point E. The center of the field line ADA of radius a lies at F while the center of the displaced flux bundle lies at F'.

pendicular to the ray path. This formal statement is nothing more than the requirement already mentioned, that the transverse gradient $\partial P/\partial s_\perp$ of the ambient pressure P is balanced by the opposing transverse magnetic tension force TK. In the present case $T = B^2/4\pi$ and $P = B^2/8\pi$, which yields (68). For the present problem, $B \sim \varpi^{-2}$ from which it follows that the ray paths are either circles $\varpi = const$ concentric about the origin, or radial lines $\varphi = const$ emanating from the origin, or circles passing through the origin $\varpi = \lambda cos(\varphi - \chi)$ (see Appendix A). Neither of the first two is appropriate, so the perturbed field lines extending between the surface of Earth ($\varpi = R$) and the ions at the equator ($\varpi = a$) must be arcs of circles that pass through the origin. The unperturbed field lines are the circles

$$\left(y - \frac{1}{2}a\right)^2 + z^2 = \frac{1}{4}a^2,$$

of course, tangent to the z axis at the origin. The perturbed field lines are not necessarily tangent to the z axis.

Now the flux bundle containing the ions is displaced outward and it is convenient to think of each point (y, z) on the unperturbed circular field line moving radially outward (from the center of the circle) to the new position $(y + \Delta y, z + \Delta z)$. For such a radial displacement,

$$\frac{\Delta z}{\Delta y} = \frac{z}{y - \frac{1}{2}a}. \tag{69}$$

The transverse radial displacement H of the flux bundle is

$$H = [(\Delta y)^2 + (\Delta z)^2]^{\frac{1}{2}}$$
$$= \Delta y \,\frac{\frac{1}{2}a}{y - \frac{1}{2}a}. \tag{70}$$

The perturbation introduced by the ions displaces the center of the circle from $y = \frac{1}{2}a, z = 0$ to some new position $y = \frac{1}{2}(a + \Delta a), z = \Delta b$. The radius of the circle becomes $\frac{1}{2}(a + \Delta c)$. Neglecting terms second order in $\Delta a, \Delta y$, etc., the equation for the displaced circle reduces to

$$2(y - \frac{1}{2}a)\Delta y + 2z\Delta z = y\Delta a + 2z\Delta b + \frac{1}{2}a(\Delta c - \Delta a)$$

for the upper ($z > 0$) half of the field line. Using (69) to eliminate Δz, the result is

$$\frac{\frac{1}{2}a^2}{y - \frac{1}{2}a}\Delta y = y\Delta a + 2z\Delta b + \tfrac{1}{2}a(\Delta c - \Delta a). \tag{71}$$

Three conditions are imposed on the perturbed circle. The first is that the circle pass through the origin, so that $\Delta y = \Delta z = 0$ where $y = z = 0$, requiring $\Delta c = \Delta a$. The second condition is that $d(y + \Delta y)/dz = -\vartheta$ as z declines to zero at the position $\varpi = a$ of the trapped ions. Since $dy/dz \sim z$ (vanishing as $z \to 0$), differentiation of (71) for Δy yields $\Delta b = -\frac{1}{2}a\vartheta$. The third condition is that the field line is fixed ($\Delta y = \Delta z = 0$) at the solid Earth, represented by

$$\varpi = R \equiv acos\varphi_R,\ \varphi \equiv \varphi_R,\ y = R^2/a,\ z = R(1 - R^2/a^2)^{\frac{1}{2}},$$

requiring that

$$\Delta a \cong \vartheta\frac{a^2}{R}\left(1 - \frac{R^2}{a^2}\right)^{\frac{1}{2}} \tag{72}$$

to the first order in ϑ, given by (63). In the limit $a >> R$, these results reduce to

$$\Delta a = \Delta c \cong a\vartheta\left(\frac{a}{R}\right)$$
$$= -2\frac{a}{R}\Delta b.$$

Thus Δa and Δc are larger than Δb by the large factor $2a/R$.

The picture is clear in the limit of large a/R, then. The circle, sketched in Figure 4, is rotated away from tangency to the z axis by the angle ϑ and the diameter of the circle is increased by Δa, so that to lowest order in R/a the equatorial ($\varphi = 0$) apex of the flux bundle is displaced outward the distance $H(0) = \Delta a$. To lowest order the center of the circle remains on the y axis, but is displaced outward a distance $\frac{1}{2}\Delta a$. In Figure 4 the unperturbed field line is a circle of diameter a with its center at the point F on the y axis. The perturbed field line has a diameter $a + \Delta a$ with center at the point F'. The line OF' is inclined at an angle ϑ to the y axis. The perturbed field line intersects the y axis at point B, at a distance $H(\varphi = 0)$ beyond point A, where it is inclined by the angle ϑ to the z direction. The two points D and D' represent the fixed terrestrial footpoints of the perturbed field line at $\varpi = R$.

Analytically, it follows from (71) that

$$\Delta y = \frac{(y - \frac{1}{2}a)}{\frac{1}{2}a^2}(y\Delta a + 2z\Delta b)$$

and from (70) that

$$H(y, z) = \frac{1}{a}(y\Delta a + 2z\Delta b) \tag{73}$$

on the upper half ($z > 0$) of the unperturbed field line ($y = \varpi^2/a, z = \varpi(1 - \varpi^2/a^2)^{\frac{1}{2}}$) so that $H(\varphi = 0) = \Delta a$ at the apex ($\varpi = y = a, z = \varphi = 0$). That is to say, the pressure of the ions causes an outward displacement Δa of the apex of the field line from $y = a$ to $y = a + \Delta a = a + H(\varphi = 0)$ where

$$H(\varphi = 0) = \vartheta\frac{a^2}{R}\left(1 - \frac{R^2}{a^2}\right)^{\frac{1}{2}}$$
$$= \frac{4\pi U}{B(a)^2}\frac{a}{R}\left(1 - \frac{R^2}{a^2}\right)^{\frac{1}{2}}. \tag{74}$$

This result is to be compared with (66) giving $H(\varphi)$ at the apex from the $\mathbf{E}, \mathbf{j}$ paradigm. The factor φ_R in (65) is replaced by the factor $(a/R)(1 - R^2/a^2)^{\frac{1}{2}}$. In the limit of large a/R, φ_R goes to $\frac{1}{2}\pi$, which is replaced by the large number a/R in the $\mathbf{B}$, $\mathbf{v}$ paradigm. The difference is qualitative in the limit of large a/R.

It is essential to understand precisely where the difference arises. Both methods provide the same inclination ϑ, given by (63) at the fixed footpoint of the field line and at the apex. The $\mathbf{E}$, $\mathbf{j}$

paradigm gives uniform ϑ everywhere along the unperturbed field line. It is easy to show that the same calculation in the present case gives quite a different result, with

$$\vartheta(\varphi) = -\frac{dH(\varphi)}{ds} = -\frac{1}{a}\frac{dH(\varphi)}{d\varphi}.$$

Noting that $y = a\, cos^2\varphi$ and $z = a\, sin\varphi\, cos\varphi$ along the unperturbed field line, it follows from (73) that

$$\vartheta(\varphi) = \frac{1}{a}[\Delta a\, sin 2\varphi - 2\Delta b\, cos\, 2\varphi]$$

or, with $\varpi = a\, cos\varphi$,

$$\vartheta(\varpi) = \vartheta(a)\left[2\left(1-\frac{R^2}{a^2}\right)^{\frac{1}{2}}\frac{\varpi}{R}\left(1-\frac{\varpi^2}{a^2}\right)^{\frac{1}{2}} + 2\frac{\varpi^2}{a^2} - 1\right]$$

with $\vartheta(a)$ representing the inclination at the apex, given by (63). It is evident that $\vartheta(R) = \vartheta(a)$, so ϑ has the same value at both ends, already noted. The difference lies in the factor $(\varpi/R)(1 - \varpi^2/a^2)^{\frac{1}{2}}$ in the brackets on the right-hand side, increasing to the large value $(2^{\frac{1}{2}}/3)a/R$ at its maximum at $\varpi/a = 1/3^{\frac{1}{2}} = 0.577$, before falling to zero at $\varpi = a$. The question arises as to how ϑ can become so large between the end points. The answer is, of course, that it is the inclination of the flux bundle relative to the ambient unperturbed field along the perturbed path that matters. The dashed curve in Figure 4 represents a field line $\varpi = a\, cos\varphi$ of the unperturbed field, passing by the perturbed flux bundle at the position E. It is obvious by inspection that the angle of intersection is of the order of $\vartheta(a)$. The formal calculation is carried through in Appendix **B** showing that the inclination of the perturbed field lines relative to the local unperturbed ambient field has the uniform value $\vartheta(a)$, given by (63), along the entire length of the perturbed field line. The error in computing the displacement of the field line through the **E**, **j** paradigm arises in placing $J_{\parallel}$ along the unperturbed field lines.

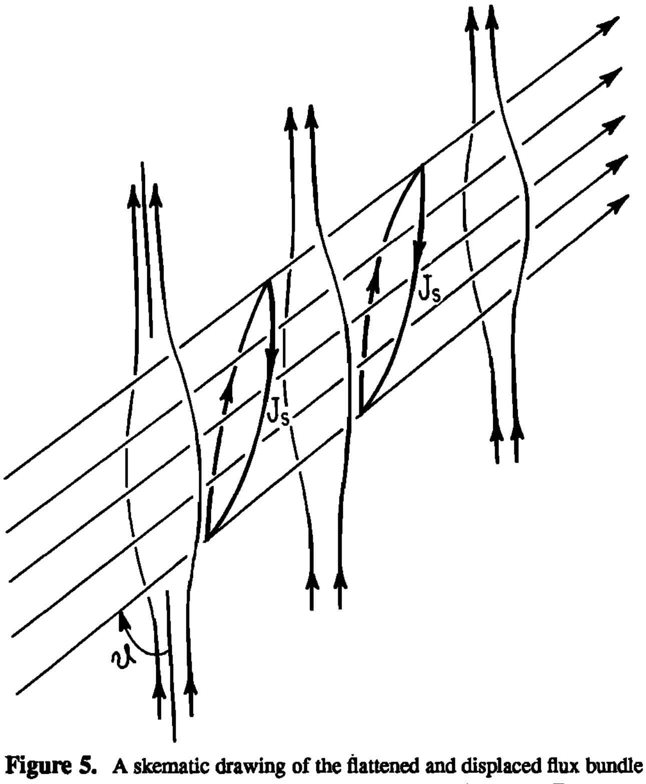

Figure 5. A skematic drawing of the flattened and displaced flux bundle passing between the field lines of the ambient field on either side. The closed path of the surface current J_s flowing around the flux bundle is indicated by the heavy lines. The direction of J_s on each side of the flux bundle is halfway between the perturbed and unperturbed magnetic field.

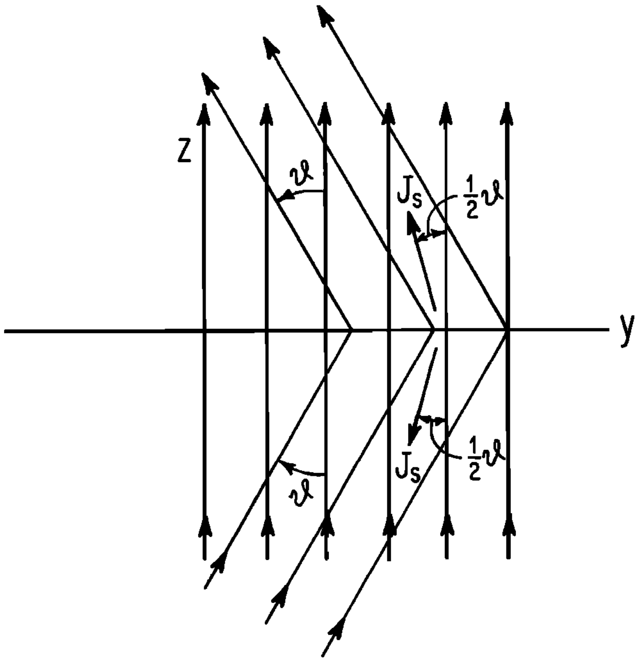

Figure 6. A drawing of the ambient and displaced fields in the neighborhood of the equatorial plane, again showing the current J_s to point halfway between the two fields.

In fact, the current pattern is qualitatively different from that employed in the **E**, **j** paradigm. We need only compute $\nabla \times \mathbf{B}$ to see how the current flows. The essential point is that the surface current J_s associated with the displaced flux bundle flows in the direction halfway between the external ambient field and the field of the displaced flux. Hence the currents close locally, as indicated in Figure 5.

To run quickly through the form of the currents, consider Figure 6 which is an idealized sketch of the displaced flux bundle inclined relative to the ambient field by the angle ϑ where it crosses the equatorial plane shown against the background ambient field B. The surface current J_s is inclined to the ambient field by $\frac{1}{2}\vartheta$. The force $\mathcal{F}$ per unit area in the equatorial plane responsible for relocating the flux bundle involves the Lorentz force JB_z/c (opposing the diamagnetic repulsion) in the equatorial plane. That is to say, the diamagnetic repulsion is opposed by the Maxwell stress $B_yB_z/4\pi$ in the flux bundle on each side of the equatorial plane. Hence

$$\frac{JB_z}{c} = 2\frac{B_yB_z}{4\pi} \tag{75}$$

from which it follows that the surface current $J(= cB/2\pi)$ flows in the equatorial plane across the field and out of the plane of the page in Figure 6. Now the surface current density J_s arising in the shear plane between two magnetic fields whose directions differ by the angle ϑ follows from Ampere's law as

$$J_s = \frac{cB}{2\pi} sin\frac{1}{2}\vartheta$$

with the component

$$\begin{aligned} J_s cos\frac{1}{2}\vartheta &= \frac{cB}{4\pi} sin\vartheta \\ &= \frac{cB_y}{4\pi} \\ &= \frac{1}{2}J \end{aligned} \tag{76}$$

perpendicular to the equatorial plane flowing away in both directions from the sheet of ions at $z = 0$. Hence current is conserved. However, it must be emphasized again that conservation of current computed from Ampere's law is nothing more than a check on the arithmetic of the computation. The fields are determined by stress balance, and Ampere's law guarantees conservation of current for any field deformation. So in place of Stern's suggestion that the region 1 Birkland currents are primarily a consequence of $\mathbf{J}_{\|}$ from energetic particles trapped in the geomagnetic tail, the $\mathbf{B}$, $\mathbf{v}$ paradigm states that region 1 currents arise simply from Ampere's law in the magnetic shear between the field of the geotail and the closed field lines at low magnetic latitudes. The magnetic field of the geotail is maintained in its extended posture by its connections into the solar wind, by intrusion of the solar wind into the magnetopause, and by the inflation of an internal population of energetic particles, each in variable and generally unknown proportions. At any moment in time the distribution of the shear and of the region 1 currents over latitude is determined by detailed combination of these several dynamical effects distributed along the geotail.

An essential result of the $\mathbf{B}$, $\mathbf{v}$ paradigm is the flattening of the displaced flux bundle by the pressure of the ambient field on each side [*Parker*, 1981a, b], shown schematically in Figure 5. The field outside the displaced flux bundle is hardly perturbed at all, whereas within the flux bundle the field all deviates by the angle ϑ from the ambient field direction. This extends down to the dissipationless ionosphere employed in the calculation. The result is a decrease of the horizontal component $B_o cos\varphi_o$ in the amount of $\vartheta B_o sin\varphi_o$ at the footpoints of the displace flux bundle, and zero elsewhere. The flattening of the displaced bundle provides a foot point in the form of a narrow north-south strip.

The flattening of the flux bundle implies that the cluster of ions at the apex in the equatorial plane is also flattened to some degree. The dynamics of the flattening of the cluster presents an interesting problem, particularly when combined with the field-aligned currents and the associated ionospheric dissipation.

6.3. Isotropic Particle Distribution

Consider briefly the problem of computing the deformation of a dipole field by various particle distributions. *Sckopke* [1966] provides a general result for a axially symmetric equilibrium shell of energetic ions with arbitrary pitch angle distribution. Here we take up the example of a single elemental flux bundle in a two dimensional magnetic dipole inflated by plasma with isotropic thermal velocity, for which the plasma pressure p is uniform along the flux bundle. We neglect the small loss cone that arises in the presence of a solid Earth at $\varpi = R$, causing the plasma pressure to fall to zero as ϖ nears R. The purpose of the example is to illustrate the utility of the optical analogy in the $\mathbf{B}$, $\mathbf{v}$ paradigm.

For the two-dimensional dipole the ambient field is again provided by (50). The magnetic field $b(\varpi)$ within the inflated flux bundle follows from the condition of pressure equilibrium across the bundle, yielding

$$b(\varpi) = B_o(R^4/\varpi^4 - \beta)^{\frac{1}{2}} \tag{77}$$

where $\beta \equiv 8\pi p/B_o^2$ represents the plasma β at the surface $\varpi = R$. The field strength $b(\varpi)$ within the inflated flux bundle provides the effective index of refraction for that field. The field lines are the geodesics defined by

$$\delta \int d\varpi \mathcal{L}(\varpi, \varphi, \varphi') = 0$$

where now

$$\mathcal{L}(\varpi, \varphi') = (1 + \varpi^2\varphi'^2)^{\frac{1}{2}}(R^4/\varpi^4 - \beta)^{\frac{1}{2}}$$

with $\varphi' \equiv d\varphi/d\varpi$. Then since $\partial\mathcal{L}/\partial\varphi = 0$, Euler's equation

$$\frac{d}{d\varpi}\left(\frac{\partial\mathcal{L}}{\partial\varphi'}\right) - \frac{\partial\mathcal{L}}{\partial\varphi} = 0$$

can be integrated to give

$$\frac{\partial\mathcal{L}}{\partial\varphi'} = 1/\kappa$$

where κ is a constant with dimensions of inverse length. This can be written as

$$\kappa\varphi'(R^4 - \beta\varpi^4)^{\frac{1}{2}} = (1 + \varpi^2\varphi'^2)^{\frac{1}{2}}. \tag{78}$$

or

$$\beta^{\frac{1}{2}}\frac{d\varphi}{d\xi}\left[(\xi_1^2 - \xi^2)(\xi_2^2 + \xi^2)\right]^{\frac{1}{2}} = \pm 1 \tag{79}$$

where $\xi \equiv \kappa\varpi$ and

$$\xi_{1,2}^2 = \frac{1}{2\beta}\left[(1 + 4n)^{\frac{1}{2}} \mp 1\right], \tag{80}$$

$$n \equiv \beta\kappa^4/R^4. \tag{81}$$

Equation (79) reduces to the quadrature

$$\beta^{\frac{1}{2}}(\varphi - \varphi_1) = \int_{\xi_1}^{\xi} \frac{d\xi}{[(\xi_1^2 - \xi^2)(\xi_2^2 + \xi^2)]^{\frac{1}{2}}} \tag{82}$$

where the integration constant φ_1 is chosen such that the apex $\xi = \xi_1$ of the path occurs at $\varphi = \varphi_1$. The elliptic integral is a standard form, from which it follows that

$$\begin{aligned} (1 + 4n)^{\frac{1}{4}}(\varphi - \varphi_1) &= F(cos^{-1}\xi/\xi_1, k), \\ &= cn^{-1}(\xi/\xi_1, k), \end{aligned} \tag{83}$$

where the modules k is

$$k = \frac{1}{2^{\frac{1}{2}}}\left[1 - \frac{1}{(1+4n)^{\frac{1}{2}}}\right]^{\frac{1}{2}}, \tag{84}$$

$$\cong n^{\frac{1}{2}}\left[1 - \frac{3}{2}n + \frac{31}{8}n^2 - \frac{187}{16}n^3 + O(n^4)\right].$$

Then

$$\xi = \xi_1 cn[(1+4n)^{\frac{1}{4}}(\varphi - \varphi_1), k].$$

Placing the apex of the field line at $\varpi = L$, where $d\varpi/d\varphi = 0$, it follows from (78) that

$$\kappa = L/R^2(1-\chi)^{\frac{1}{2}} \tag{85}$$

where $\chi = \beta L^4/R^4 < 1$. Note that if $\chi > 1$, the plasma pressure at the apex exceeds the ambient pressure and there is no equilibrium solution, except for radial lines ($d\varphi/d\varpi = 0$).

It is sufficient for present purposes to treat the case of weak inflation, $\chi << 1$, for which it is convenient to note the exact relations $\beta\kappa^2 L^2 = \chi/(1-\chi), \kappa^2 = \chi/(1+\chi), \xi_1^2 = \beta\chi/(1-\chi)$, $\xi_2^2 = \beta/(1-\chi)$, $n = \chi/(1-\chi)^2, (1+4n)^{\frac{1}{2}} = (1+\chi)/(1-\chi)$. For a field line with apex at the equator $\varphi = \varphi_1 = 0$, the field line intersects $\varpi = R$ at the latitude φ_R given by (83) as

$$\varphi_R = \frac{F(cos^{-1}R/L, k)}{(1+4n)^{\frac{1}{4}}} \tag{86}$$

$$\cong cos^{-1}\frac{R}{L} - \frac{1}{4}\chi\left[3cos^{-1}\frac{r}{L} + \frac{R}{L}(1 - \frac{R^2}{L^2})^{\frac{1}{2}}\right] \tag{87}$$

$$+O(\xi^2)$$

The field line with the same apex radius L in the absence of inflation reaches $\varpi = R$ at the latitude $cos^{-1}R/L$. from which it is clear that the foot points of the inflated field lines have a latitude that is lower by the amount

$$\Delta\varphi_R \cong \frac{1}{4}\chi\left[3cos^{-1}\frac{R}{L} + \frac{R}{L}\left(1 - \frac{R^2}{L^2}\right)^{\frac{1}{2}}\right] \tag{88}$$

than the ambient field lines. That is to say, the inflated field lines are distended, as we would expect. This is best illustrated by computing the change in the radial distance to the apex as a consequence of a small inflation β or χ while φ_R remains fixed. Then to lowest order $n \cong k^2 \cong \chi$. Use (87) to compute $\partial L/\partial\chi$, with the result that L changes by the small amount H, given by

$$\frac{H}{L} = \frac{L}{4R}\left(1 - \frac{R^2}{L^2}\right)^{\frac{1}{2}}\left[3\,cos^{-1}\frac{R}{L} + \frac{R}{L}\left(1 - \frac{R^2}{L^2}\right)^{\frac{1}{2}}\right]\chi,$$

$$= \frac{1}{4}\beta\left(\frac{L}{R}\right)^5\left(1 - \frac{R^2}{L^2}\right)^{\frac{1}{2}}\left[3\,cos^{-1}\frac{R}{L} + \frac{R}{L}\left(1 - \frac{R^2}{L^2}\right)^{\frac{1}{2}}\right]$$

to first order in χ.

In order to compare this result with (74) ($L \cong a$), note that the thermal energy density of a monatomic gas is $\frac{3}{2}p$ with $p = \beta B_o^2/8\pi = \beta[B^2(L)/8\pi]\,L^4/R^4$. The area between the field lines separated by unit distance at the equator can be computed to first order in χ from the unperturbed field, for which the separation $H(\varphi)$ of the lines is proportional to y/L, where $y = L\,cos^2\varphi$ along the field line. Then with arc length $ds = Ld\varphi$, it is readily shown that the area is $L(\varphi_R + sin\varphi_R cos\varphi_R) \sim L[cos^{-1}R/L + (R/L)(1 - R^2/L^2)^{\frac{1}{2}}]$. It follows that the thermal energy U in the flux bundle with unit cross sectional area in the equatorial plane ($\varphi = 0$) is

$$U = \frac{3}{2}pL[cos^{-1}\frac{R}{L} + \frac{R}{L}(1 - \frac{R^2}{L^2})^{\frac{1}{2}}].$$

Using this relation to express p or β in terms of U, it follows that

$$\beta = \frac{16\pi U}{3LB_o^2[cos^{-1}R/L + (R/L)(1 - R^2/L^2)^{\frac{1}{2}}]},$$

$$\frac{H}{L} = \frac{4\pi L^4 U[cos^{-1}R/L + \frac{1}{3}(R/L)(1 - R^2/L^2)^{\frac{1}{2}}]}{B_o^2 R^5[cos^{-1}R/L + (R/L)(1 - R^2/L^2)^{\frac{1}{2}}]}$$

where B_o is the field intensity $B(R)$ at $\varpi = R$. In terms of the field B(L),

$$H = \frac{4\pi UL}{B(L)^2 R}\left[\frac{cos^{-1}R/L + \frac{1}{3}(R/L)(1 - R^2/L^2)^{\frac{1}{2}}}{cos^{-1}R/L + (R/L)(1 - R^2/L^2)^{\frac{1}{2}}}\right].$$

In the limit of small R/L,

$$\beta \cong \frac{32U}{3LB_o^2},$$

$$H \cong \frac{4\pi U}{B(L)^2}\frac{L}{R}.$$

This is the same as (74). That is to say, inflation of a flux bundle with an isotropic plasma of energy U causes the apex of the inflated flux bundle to move outward by about the same distance as the same energy in particles with pitch angle $\frac{1}{2}\pi$ at the equatorial plane.

It should be noted again, that the displaced flux bundle tends to flatten from the pressure of the ambient field across which it extends. The foot point is again a narrow north-south strip.

7. Flux Transfer

7.1. Individual Flux Bundles

The transfer of geomagnetic flux bundles into the geomagnetic tail, following reconnection with the interplanetary magnetic field at the sunward magnetopause, is an integral part of magnetospheric convection at times of geomagnetic substorms and storms. Relatively little attention has been paid to the dynamics of the individual flux bundles in the process of transfer. This section treats the subject briefly to show how the **B**, **v** paradigm handles the problem. As we shall see, the basic concepts are clear enough whereas the details of the phenomenon are sufficiently chaotic as to be quantitatively intractable.

Consider a flux transfer event, involving the pick up of a magnetospheric flux bundle from the sunward magnetopause from where it is carried in the solar wind in the magnetosheath back into the geotail. A typical flux transfer bundle is observed to have a diameter of the order of $1R_E$ and an internal magnetic field of the order of 10^{-4} G, from which one infers a total magnetic flux of the order of 4×10^{13} Maxwells. The flux bundle connects into an ionospheric foot point with an area of 7×10^{13} cm ^{2} (for $B = 0.6$ G), equivalent to a square 80 km on a side. The outer end of the flux bundle at the magnetopause is transported a distance of the order of 2×10^{10} cm ($30R_E$) from the sunward magnetopause into the geotail in the 200 km/s solar wind in the magnetosheath in a time of the order of 10^3 s. The great mass of the ionospheric foot points of the flux bundle suggests that the foot points do not move significantly in that time. On the other hand, the Alfven speed in the outer magnetosphere is of the order of 10^3 km/s, as a consequence of the low ambient plasma density ($N \leq 1$ atom/cm ^{3}) beyond the plasmapause. The relatively slow transport velocity of 200 km/s at the magnetopause suggests that the displaced flux bundle is not far from quasi-static equilibrium within the outer magnetosphere during the transport into the geotail. Hence, except near the ionosphere, the instantaneous path of a moving flux bundle can be approximated by the static equilibrium path between the outer end, moving with the solar wind, and the fixed ionospheric foot point. In the simple case of a two-dimensional model of the geomagnetic dipole, treated in subsections 6.2 and 6.3, it follows from the optical analogy that the flux bundle lies along a circle that intersects the origin. Denoting the ionospheric foot point by $y = a, z = b$ and the outer end at the magnetopause by $y = Y(t), z = Z(t)$, it is readily shown that the path of the bundle is given by the circle

$$(y^2 + z^2)(bY - aZ) + y[a^2 Z - bY^2 + bZ(B - Z)]$$
$$-z[b^2Y - aZ^2 + aY(a - Y)] = 0 \qquad (89)$$

at any instant in time.

The motion $\mathbf{u}$ of the displaced flux bundle at any point determines the local electric field $\mathbf{E} = - \mathbf{u} \times B/c$ within the flux bundle. The surface current density around the displaced flux bundle is readily computed from the inclination of the displaced bundle relative to the ambient magnetic field. In particular, it follows from (89) that

$$\frac{dz}{dy} = \frac{a^2 Z - bY^2 + bZ(b - Z) + 2(bY - aZ)y}{b^2Y - aZ^2 + aY(a - Y) - 2(bY - aZ)z} \qquad (90)$$

along the path of the displaced flux bundle. For the ambient field line with an apex at $y = L, z = 0$, we have

$$y^2 - yL + z^2 = 0$$

$$\frac{dz}{dy} = \frac{L - 2y}{2z}.$$

Hence, for the ambient field line through the point (y, z) it follows that

$$L = \frac{y^2 + z^2}{y}$$

$$\frac{dz}{dy} = \frac{z^2 - y^2}{2yz} \qquad (91)$$

at that point. The angle between the displaced flux bundle and the ambient field is

$$\vartheta(y, z) = tan^{-1}\left(\frac{dz}{dy}\right)_d - tan^{-1}\left(\frac{dz}{dy}\right)_a \qquad (92)$$

where $(dz/dy)_d$ is the slope of the displaced bundle given by (90) and $(dz/dy)_a$ is the slope of the ambient field given by (91). The electric current pattern around and along the displaced flux bundle is described in subsection 6.2 and is illustrated in Figure 5. The inability of the $\mathbf{E}$, $\mathbf{j}$ paradigm to address the problem of finding the path of a widely displaced flux bundle is obvious. Note in particular that the motion of the lower end of the flux bundle is determined largely by the mechanical motion of the ionosphere and has no direct connection to the electric field in the rapidly moving outer end of the flux bundle. As noted earlier, the force exerted on the ionosphere by the displaced flux bundle is proportional to the displacement of the outer end of the flux bundle rather than to the instantaneous rate of displacement to which the electric field is related, illustrated in subsection 7.2.

It should be noted that the basic physical principles are the same in a three-dimensional dipole, but the computation is much more complicated. First, the Euler equation for the equilibrium path is of the form of Abel's equation and has no elementary solutions. Second, the outer end of the displaced flux bundle at the magnetopause is transported by the solar wind. In two dimensions this is parallel to the meridional planes in which the field lines lie. However, in three dimensions the displacement generally has a nonvanishing component perpendicular to the local geomagnetic field at the magnetopause. The flux bundle slides freely only in the direction parallel to the geomagnetic field, but not across the geomagnetic field, so the motion perpendicular to the geomagnetic field strongly deforms the geomagnetic field at the magnetopause, probably resulting in local reconnection and generally creating a messy theoretical problem. The essential point is that the problem can be stated easily in terms of the $\mathbf{B}$, $\mathbf{v}$ paradigm even if the dynamics is too complicated to allow a formal solution.

7.2. Ionospheric Motion

Now by the time a large body of displaced magnetic flux accumulates in the geotail (as during the onset of a substorm), the ionosphere begins to move and it is of interest to note the formulation of this dynamical problem. The succeeding sections treat the detailed problem, recognizing the partially ionized state of the ionosphere. However, as preparation for the complications of the Hall and Pedersen conductivities, it is useful to treat an idealized case to enumerate the elementary stress balance and momentum transfer. There is nothing particularly new here except the emphasis on the direct approach to the dynamics, describing the momentum in terms of the plasma velocity and the stress in terms of the magnetic field.

Consider, then, the simple idealized plane polar ionosphere and magnetosphere model neglecting all resistivity and viscosity. Imagine an ionosphere of uniform density ρ confined between the planes $z = 0$ and $z = L$. Consider the idealized case in which the nonconducting atmosphere below ($z < 0$) has no viscosity so the ionosphere slides freely about except for its own inertia. The magnetosphere in the space $L < z < \Lambda(\Lambda >> L)$ has negligible material density and essentially infinite Alfven speed. The whole system is threaded by a uniform vertical magnetic field B and the magnetopause at $z = \Lambda$ is subject to arbitrary horizontal displacement, as when the field is picked up by the solar wind and transported a large distance H along the magnetopause in, say, the y direction. The layout is sketched in Figure 7.

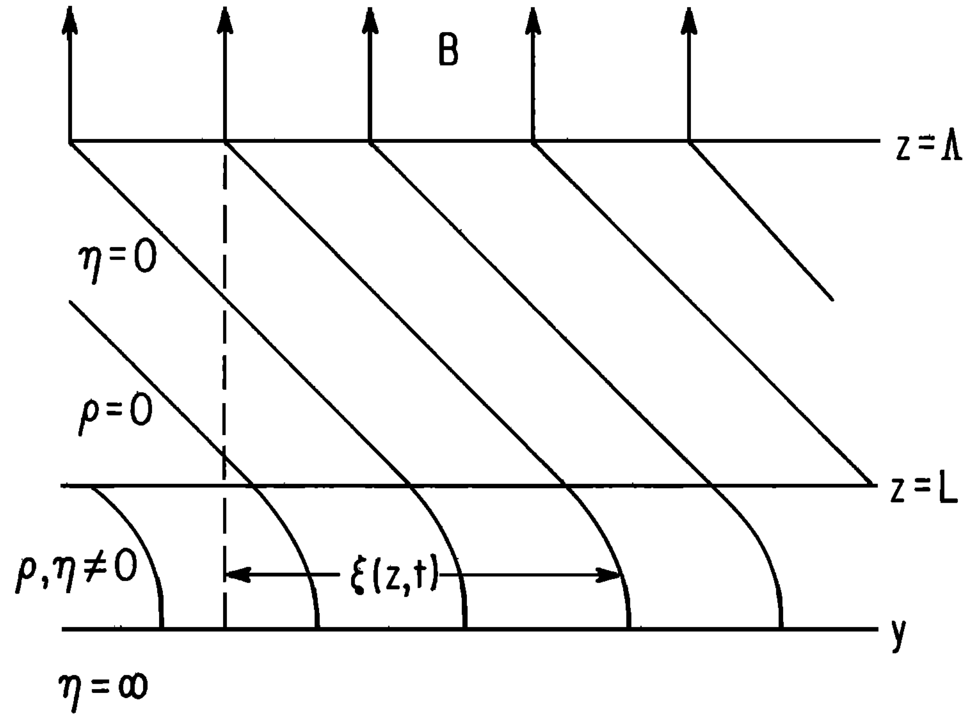

Figure 7. A schematic drawing of the sheared field (0, Bb (z,t),B) associated with the displacement ξ(z,t) relative to the footprint at the magnetopause $z = \Lambda$.

Denote the Lagrangian displacement of an element of fluid in the ionosphere by $\xi(z,t)$, assuming that $\partial/\partial y = 0$. The fluid velocity $v(\xi,t)$ in the ionosphere is $\partial\xi/\partial t$, introducing the magnetic field $Bb(z,t)$ in the y direction at the rate described by the induction equation

$$\begin{aligned}\frac{\partial b}{\partial t} &= \frac{\partial v}{\partial z}, \\ &= \frac{\partial^2 \xi}{\partial z \partial t}.\end{aligned} \tag{93}$$

In the present idealized example of a vertical field without local compression the displacements are horizontal and uniform over the horizontal plane so that they produce no magnetic perturbation below the ionosphere. The appropriate boundary condition for $b(z,t)$ is $b(0,t) = 0$ with $b = 0$ throughout the nonconducting atmosphere in $z < 0$. The conducting Earth plays a role only where $\partial/\partial y \neq 0$.

The Maxwell stress $M_{yz} = Bb/4\pi$ reacts back on the fluid for which the momentum equation is

$$\rho\frac{\partial^2 \xi}{\partial t^2} = \frac{B^2}{4\pi}\frac{\partial b}{\partial z} \tag{94}$$

The Alfven speed $B/(4\pi\rho)^{\frac{1}{2}}$ in the ionosphere is denoted by C. The Alfven speed is essentially infinite in the tenuous magnetosphere ($L < z < \Lambda$), so that if the field is held fixed at the magnetopause ($z = \Lambda$), it is convenient to measure the displacement ξ from the position at which the field is vertical ($b = 0$). Hence,

$$\begin{aligned}b(L,t) &= -\frac{\xi(L,t)}{\Lambda - L} \\ &\cong -\frac{1}{\Lambda}\xi(L,t)\end{aligned} \tag{95}$$

and $b(z,t) = b(L,t)$ throughout $L < z < \Lambda$. Equation (93) can be integrated once to give

$$b(z,t) = \partial\xi/\partial z \tag{96}$$

in the ionosphere, where the integration constant is zero with the choice $\xi = 0$ for $b = 0$. Substituting into (94) gives

$$\frac{\partial^2 \xi}{\partial t^2} - C^2\frac{\partial^2 \xi}{\partial z^2} = 0. \tag{97}$$

Consider the solution

$$\xi = D cos\frac{\omega z}{C} cos\omega t, \tag{98}$$

where D is an arbitrary constant. It follows from (96) that

$$b = D\frac{\omega}{C} sin\frac{\omega z}{C} cos\omega t. \tag{99}$$

Substituting (98) and (99) into (95), the result is

$$\frac{\omega L}{C} tan\frac{\omega L}{C} = \frac{L}{\Lambda}. \tag{100}$$

It is clear that in the limit of small L/Λ, the quantity $\omega L/C$ is either small compared to one, so that

$$\omega \cong \frac{C}{(L\Lambda)^{\frac{1}{2}}}, \tag{101}$$

or

$$\omega \cong \pm n\pi\left(1 + \frac{L}{n^2\pi^2\Lambda} + \cdots\right)$$

where $n = 1, 2, 3, \cdots$ We are concerned here only with the fundamental frequency, given by (101).

Suppose, then, that the time $t = 0$ finds the magnetopause ($z = \Lambda$) displaced a distance H in the negative y - direction and thereafter held fixed. The system is without motion at the time $t = 0$, but the nonvanishing Maxwell stress accelerates the ionosphere from rest. The initial value of ξ is then H throughout the ionosphere and there is a kink in the field at $z = L$, with a deflection $tan^{-1}H/\Lambda$. The kink provides an initial sharp transient, of no particular physical interest, that propagates back and forth between $z = 0$ and $z = \Lambda$ with partial reflection at each crossing of $z = L$ so that the initial pulse rapidly degrades into an increasing number of pulses of diminishing amplitude. The principal motion is the fundamental oscillation

$$\xi(z,t) = H cos \frac{z}{(L\Lambda)^{\frac{1}{2}}} cos \frac{Ct}{(L\Lambda)^{\frac{1}{2}}}$$

$$\cong H cos \frac{Ct}{(L\Lambda)^{\frac{1}{2}}}$$

$$b(z,t) = -\frac{H}{(L\Lambda)^{\frac{1}{2}}} sin \frac{z}{(L\Lambda)^{\frac{1}{2}}} cos \frac{Ct}{(L\Lambda)^{\frac{1}{2}}}$$

$$\cong -\frac{Hz}{L\Lambda} cos \frac{Ct}{(L\Lambda)^{\frac{1}{2}}}$$

in the ionosphere, neglecting terms $O(L/\Lambda)$ compared to one. It follows that

$$v(t) \cong -\frac{CH}{(L\Lambda)^{\frac{1}{2}}} sin \frac{Ct}{(L\Lambda)^{\frac{1}{2}}}$$

The electric current and electric field are uniform across the ionosphere with the instantaneous values

$$\mathbf{j}(t) = -\mathbf{e}_x \frac{cB}{4\pi} \frac{\partial b}{\partial z}$$

$$\cong +\mathbf{e}_x \frac{cBH}{4\pi L\Lambda} cos \frac{Ct}{(L\Lambda)^{\frac{1}{2}}}$$

$$\mathbf{E}(t) = -\mathbf{e}_x \frac{vB}{c}$$

$$\cong +\mathbf{e}_x \frac{BCH}{c(L\Lambda)^{\frac{1}{2}}} sin \frac{Ct}{(L\Lambda)^{\frac{1}{2}}}$$

in the ionosphere $(0 < z < L)$. In the magnetosphere b is uniform with the value $b(L,t)$ given above, while the velocity is

$$v(z,t) = v(t)(\Lambda - z)/(\Lambda - L).$$

The electric field is accordingly

$$E(z,t) = E(t)\,(\Lambda - z)/(\Lambda - L),$$

vanishing at the magnetopause $z = \Lambda$, from which it is clear that the electric field at the magnetopause has nothing to do with the motion within the magnetopause and in the ionosphere. The initial, relatively rapid displacement of the magnetopause by a distance H in the negative y direction at time $t = 0$ can be adequately described by the velocity $-H\delta(t)$ at $z = \Lambda$ with

$$v(z,t) = -H\delta(t)(z - L)/(\Lambda - L)$$

throughout the magnetosphere. The associated electric field is the short intense pulse

$$\mathbf{E}(z,t) = e_x H\delta(t)(z - L)/(\Lambda - L)$$

which vanishes before the ionosphere has begun to move. So again, the electric field at the magnetopause is not related to the ionospheric motion and vice versa. The ionosphere moves in response to the Maxwell stresses.

It is easy to extend the calculations to include viscosity in the ionosphere and in the non-ionized atmosphere ($z < 0$) below the ionosphere, as well as a conducting Earth at $z - \ell$ etc. However, the foregoing idealized calculation is adequate for purposes of illustrating the basic principles. We turn next to a formulation of the $\mathbf{B}$, $\mathbf{v}$ paradigm in the ionosphere, taking account of the massive neutral atomic constituent.

8. Partially Ionized Gases

Consider a plane stratified ionosphere composed of N neutral atoms per cm^3, and n electrons and ions per cm^3, the ions being singly charged. The mean effective collision time for the individual ion with the ambient neutral atoms is denoted by τ_i and the collision time for an electron with the neutral atoms is τ_e. The ion-electron collisions provide an equivalent collision time τ. The electron-electron and ion-ion collision times are comparable to each other but of no particular interest. The local mean velocity of the ion gas is denoted by $\mathbf{w}$ and of the electron gas by $\mathbf{u}$, with the $\mathbf{v}$ representing the local mean velocity of the neutral atoms. The collisions provide a mean drag force on each constituent, so that for cold electrons and singly charged ions, the momentum equations are

$$M\frac{d\mathbf{w}}{dt} =$$

$$+e\left(\mathbf{E}+\frac{\mathbf{w}\times\mathbf{B}}{c}\right) - \frac{M(\mathbf{w}-\mathbf{v})}{\tau_i} - \frac{m(\mathbf{w}-\mathbf{u})}{\tau}, \quad (102)$$

$$M\frac{d\mathbf{u}}{dt} = -e\left(\mathbf{E}+\frac{\mathbf{u}\times\mathbf{B}}{c}\right) - \frac{m(\mathbf{u}-\mathbf{v})}{\tau_e} + \frac{m(\mathbf{w}-\mathbf{u})}{\tau}. \quad (103)$$

The equation of motion for the neutral atoms is

$$NM\frac{d\mathbf{v}}{dt} = -\nabla p + \frac{nM(\mathbf{w}-\mathbf{v})}{\tau_i} + \frac{nm(\mathbf{u}-\mathbf{v})}{\tau_e} \quad (104)$$

where p is the pressure. Viscous and gravitational terms can be added if desired.

Note that the electric current density $\mathbf{j}$ is

$$\mathbf{j} = ne(\mathbf{w} - \mathbf{u}) \quad (105)$$

$$= \frac{c}{4\pi}\nabla\times\mathbf{B}, \quad (106)$$

the second equality being Ampere's law. The relation can be solved for $\mathbf{u}$, yielding

$$\mathbf{u} = \mathbf{w} - \frac{c}{4\pi ne}\nabla\times\mathbf{B}. \quad (107)$$

Consider, then, the case of weak ionization $n << N$, appropriate for the terrestrial ionosphere. The inertia of the ions and electrons is small compared to the inertia of the neutral gas. The Maxwell stresses are exerted on the ions and electrons and are transferred to the neutral gas through collisions. The relatively massive neutral gas responds only sluggishly to the collisions, so that on that timescale the left-hand sides of (102) and (103) are negligible, that is, the inertia of the electrons and ions is negligible compared to the inertia of the neutral gas. The ions and electrons are in quasi-static equilibrium between the Maxwell stresses and the collisional drag of the neutral gas.

Neglect the left-hand sides add (102) and (103) and then multiply by n, with the result that

$$\frac{\mathbf{j}\times\mathbf{B}}{c}=\frac{nM(\mathbf{w}-\mathbf{v})}{\tau_i}+\frac{nm(\mathbf{u}-\mathbf{v})}{\tau_e} \tag{108}$$

The right-hand side of this relation represents the friction terms on the right-hand side of (104) with the result, using Ampere's law, that

$$NM\frac{d\mathbf{v}}{dt}=-\nabla p+\frac{(\nabla\times\mathbf{B})\times B}{4\pi} \tag{109}$$

plus whatever viscosity and gravitation effects are appropriate. This, of course, is the usual MHD momentum equation, with the fluid pushed around by ∇p and by the Lorentz force, that is, by the divergence of the Maxwell stress tensor.

Next solve (107) and (108) for $\mathbf{u}$ and $\mathbf{w}$ in terms of $\mathbf{v}, \nabla\times\mathbf{B}$, etc., with the result

$$\mathbf{w}=v+\frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi nQ}+\frac{cm/\tau_e}{4\pi neQ}\nabla\times\mathbf{B} \tag{110}$$

$$\mathbf{u}=v+\frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi nQ}-\frac{cM/\tau_i}{4\pi neQ}\nabla\times\mathbf{B} \tag{111}$$

where

$$Q\equiv\frac{M}{\tau_i}+\frac{m}{\tau_e} \tag{112}$$

It is then a straightforward calculation to solve either (102) or (103) for $\mathbf{E}$, obtaining

$$\mathbf{E}=-\frac{\mathbf{v}\times B}{c}-\frac{[(\nabla\times\mathbf{B})\times\mathbf{B}]\times\mathbf{B}}{4\pi ncQ}+$$
$$\frac{M/\tau_i-m/\tau_e}{4\pi neQ}(\nabla\times\mathbf{B})\times\mathbf{B} \tag{113}$$
$$+\frac{c}{4\pi ne^2}\left[\frac{(M/\tau_i)(m/\tau_e)}{Q}+\frac{m}{\tau}\right]\nabla\times\mathbf{B}.$$

It is convenient to define the diffusion coefficients α, β and η, with

$$\alpha\equiv\frac{cB(M/\tau_i-m/\tau_e)}{4\pi ne(M/\tau_i+m/\tau_e)} \tag{114}$$

representing the Hall resistive diffusion coefficient,

$$\beta\equiv\frac{B^2}{4\pi n\,(M/\tau_i+m/\tau_e)} \tag{115}$$

as the Pedersen resistive diffusion coefficient, and

$$\eta\equiv\frac{c^2}{4\pi ne^2}\left[\frac{(M/\tau_i)(m/\tau_e)}{M/\tau_i+m/\tau_e}+\frac{m}{\tau}\right] \tag{116}$$

as the Ohmic resistive diffusion coefficient, where B is a characteristic magnetic field strength. *Fejer* [1953] and others provide conductivities as a function of altitude in the ionosphere. Write $\mathbf{b}=\mathbf{B}/B$, so that

$$\mathbf{E}=\frac{B}{c}\{-\mathbf{v}\times\mathbf{b}$$
$$-\beta[(\nabla\times\mathbf{b})\times\mathbf{b}]\times\mathbf{b}+\alpha(\nabla\times\mathbf{b})\times\mathbf{b}+\eta\nabla\times\mathbf{b}\} \tag{117}$$

$$=\frac{B}{c}\{-\mathbf{v}\times\mathbf{b}$$
$$+\alpha(\nabla\times\mathbf{b})_\perp\times\mathbf{b}+(\eta+b^2\beta)(\nabla\times\mathbf{b})_\perp+\eta(\nabla\times\mathbf{b})_{||}\} \tag{118}$$

where the subscripts $\perp$ and $||$ refer to the components perpendicular and parallel to $\mathbf{b}$.

Note that (117) can be written

$$\mathbf{E}'=\frac{B}{c}$$
$$\times\{-\beta[(\nabla\times\mathbf{b})\times b]\times b+\alpha(\nabla\times\mathbf{b})\times b+\eta\nabla\times\mathbf{b}\} \tag{119}$$

where

$$\mathbf{E}'=\mathbf{E}+(B/c)\mathbf{v}\times b$$

is the electric field in the frame of reference of the moving neutral atoms. Equation (119) is the form of Ohm's law for a partially ionized gas, of course, but it cannot be used effectively until the velocity $\mathbf{v}$ has been determined from the dynamical equations. This difficulty is sometimes over looked in applications of the $\mathbf{E}$, $\mathbf{j}$ paradigm. Illustrative examples are provided in sections 9-11.

Substituting (117) and (118) into

$$\frac{\partial\mathbf{B}}{\partial t}=-c\nabla\times\mathbf{E}$$

yields the induction equation

$$\frac{\partial\mathbf{b}}{\partial t}=\nabla\times(\mathbf{v}\times b)$$
$$+\nabla\times\{\beta[(\nabla\times\mathbf{b})\times\mathbf{b}]\times\mathbf{b}-\alpha(\nabla\times\mathbf{b})\times\mathbf{b}-\eta\nabla\times\mathbf{b}\} \tag{120}$$
$$=\nabla\times(\mathbf{v}\times\mathbf{b})-\nabla$$
$$\times\{(\eta+b^2\beta)(\nabla\times\mathbf{b})_\perp+\alpha(\nabla\times\mathbf{b})_\perp\times\mathbf{b}+\eta(\nabla\times\mathbf{b})_{||}\}. \tag{121}$$

It is not without interest to express $\mathbf{E}$ in terms of the ion drift velocity $\mathbf{w}$ or the electron drift velocity $\mathbf{u}$. Solving (110) and (111) for $\mathbf{v}$ and substituting into (113) yields the two expressions

$$\mathbf{E}=\frac{B}{c}\{-\mathbf{w}\times\mathbf{b}+\frac{cBM/\tau_i}{4\pi neQ}(\nabla\times\mathbf{b})\times\mathbf{b}+\eta\nabla\times\mathbf{b}\} \tag{122}$$

$$\mathbf{E}=\frac{B}{c}\{-\mathbf{u}\times\mathbf{b}-\frac{cBm/\tau_e}{4\pi neQ}(\nabla\times\mathbf{b})\times\mathbf{b}+\eta\nabla\times\mathbf{b}\} \tag{123}$$

so that the induction equation can be expressed accordingly.

The term $\nabla\times(\mathbf{v}\times\mathbf{b})$ on the right-hand side of (120) represents the transport of field with the moving fluid, that is, the standard MHD induction effect. The remaining terms represent diffusion and dissipation and the Hall effect. It is instructive to write out the energy equation to see the nature of the dissipation. Forming the scalar produce of $\mathbf{b}$ with (118) and using the vector identity

$$\mathbf{A}\cdot\nabla\times\mathbf{D}=\mathbf{D}\cdot\nabla\times\mathbf{A}+\nabla\cdot(\mathbf{D}\times A),$$

it can be shown that

$$\frac{\partial}{\partial t}\frac{b^2}{8\pi}+\nabla\cdot(\mathbf{v}_\perp\frac{b^2}{4\pi})$$
$$=-\mathbf{f}\cdot\mathbf{v}-4\pi\beta f^2-\eta(\nabla\times\mathbf{b})^2/4\pi \tag{124}$$
$$-\nabla\cdot[\beta b^2\mathbf{f}+\alpha\mathbf{f}\times\mathbf{b}+\eta\mathbf{f}],$$

where $\mathbf{f}$ is the Lorentz force divided by B^2,

$$\mathbf{f} = \frac{(\nabla \times \mathbf{b}) \times \mathbf{b}}{4\pi} \tag{125}$$

The convective transport of magnetic enthalpy is $\mathbf{v}_\perp b^2/4\pi$. The first term on the right-hand side is the rate at which the fluid does work on the field. The third term is the usual Joule dissipation term, which can be written j^2/σ if desired. The second term is the Pedersen dissipation. The final term, involving the divergence of $\beta b^2\mathbf{f} + \alpha\mathbf{f} \times \mathbf{b} + \eta\mathbf{f}$ represents only a redistribution of magnetic energy within the region. It provides no net dissipation, as is obvious from the fact that it vanishes when integrated over a volume sufficiently large that $\mathbf{f}$ vanishes on the surface. The Hall effect provides no net dissipation, of course, although it provides an energy redistribution flux

$$\alpha\mathbf{f} \times \mathbf{b} = -\alpha b^2(\nabla \times \mathbf{b})_\perp \tag{126}$$

within the region of nonvanishing Lorentz force. The momentum (109) can be rewritten

$$NM\frac{d\mathbf{v}}{dt} = -\nabla p + B^2\mathbf{f} \tag{127}$$

to which viscous terms, etc. may be added if needed.

Simultaneous solution of (120) and (127), along with some statement on the pressure p, provides a complete description of the macroscopic behavior of the ionosphere. In fact, the dynamics of the ionosphere involves only small perturbations $\Delta\mathbf{B}$ of the quiet day field so that the equations can be linearized to good approximation. The linear form is studied briefly in the next section. The essential point is that, with the basic equations formulated in terms of the variables $\mathbf{v}$ and $\mathbf{B}$, the time dependent case can be treated directly. Electric fields can be calculated after the fact if they are needed. Charge conservation is automatic, and electric currents take care of themselves.

The momentum equation (127) is the familiar momentum equation of magnetohydrodynamics. The induction equation (120) is similar to the induction equation of MHD. It is evident from inspection of the right-hand side of (121) that the total resistive diffusion coefficient of $(\nabla \times \mathbf{b})_\parallel$ (representing diffusion of field perpendicular of $\mathbf{b}$) is η, while the coefficient of $(\nabla \times \mathbf{b})_\perp$ is $\eta + b^2\beta$. So the resistive diffusion is anisotropic, but still of the nature of resistive diffusion. The Hall term, whose coefficient is α, is more interesting. Neglecting resistive diffusion, write (120) as

$$\frac{\partial\mathbf{b}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{b}) - 4\pi\nabla \times \alpha\mathbf{f}, \tag{128}$$

where $\mathbf{f}$ is the dimensionless Lorentz force defined by (125). Then write the momentum equation (127) as

$$\frac{\partial\mathbf{v}}{\partial t} + \omega \times \mathbf{v} = -\frac{1}{NM}\nabla p - \nabla\frac{1}{2}v^2 + 4\pi C^2\mathbf{f}, \tag{129}$$

where $\omega = \nabla \times \mathbf{v}$ is the vorticity and $C = B/(4\pi NM)^{\frac{1}{2}}$ is the Alfven speed in the characteristic field B. Consider the idealized situation in which the neutral gas density NM and the Hall coefficient α are locally uniform. Then the curl of the momentum equation yields the vorticity equation:

$$\frac{\partial\omega}{\delta t} = \nabla \times (\mathbf{v} \times \omega) + 4\pi C^2\nabla \times \mathbf{f}. \tag{130}$$

Using this expression to eliminate $\nabla \times \mathbf{f}$ from (128) yields

$$\frac{\partial}{\partial t}\left(\mathbf{b} + \frac{\alpha}{C^2}\omega\right) = \nabla \times \left[\mathbf{v} \times \left(\mathbf{b} + \frac{\alpha}{C^2}\omega\right)\right]. \tag{131}$$

It follows that the vector $\mathbf{b} + \alpha\omega/C^2$ is carried bodily with the fluid motion $\mathbf{v}$, as distinct from the transport of $\mathbf{b}$ with $\mathbf{v}$ in the absence of the Hall effect. So the concept of a "frozen in" field remains. The field is neither $\mathbf{b}$, that is "frozen in" in the absence of the Hall resistivity, nor the vorticity in the absence of a magnetic field, but a linear combination $\mathbf{b} + (\alpha/C^2)\omega$ of the two.

Note, then, that since $m/\tau_e << M/\tau_i$ in the ionosphere, (114) yields $\alpha \cong eB/4\pi ne$, where n represents the ion and electron number densities. It follows that the coefficient α is given by

$$\alpha/C^2 \cong N/n\Omega_i \tag{132}$$

where $\Omega_i = eB/Mc$ represents the ion cyclotron frequency in the characteristic field B. Thus, with $|\omega| \sim 1/t_\omega$, where t_ω is the characteristic rotation or shearing time of the fluid motion, it follows that the ratio of $\alpha\omega/C^2$ to $\mathbf{b}$ is of the order of $(N/n)/\Omega_i t_\omega$. The ratio N/n is large ($\sim 10^3$ at an altitude of 300 km) and $\Omega_i \sim 3 \times 10^2$/sec for a single ionized oxygen atom in 0.6 G. So the ratio $\alpha\omega/C^2 b$ is small for motions with characteristic time large compared to say, 10s, and it is $\mathbf{b}$ that is tied to $\mathbf{v}$ to a first approximation. As N/n declines upward into the magnetosphere, the ratio becomes smaller, and the frozen in condition obtains (in the absence of significant diffusion). However, ideal or not, the induction equation shows that $\mathbf{b} + \alpha\omega/C^2$ is a physical concept that is useful in treating dynamics.

As is well known, the pertinent quantity is the ratio of the Pedersen and Hall coefficients β/α. To obtain some idea of the magnitude of β and α in the ionosphere, recall that the mean free paths for ions and electrons are comparable in order of magnitude. Hence $\tau_i/\tau_e \sim (M/m)^{\frac{1}{2}}$ and m/τ_e is smaller than M/τ_i by a factor of the order of $(m/M)^{\frac{1}{2}}$. It follows from (114) and (120) that

$$\begin{aligned} \alpha &\cong \frac{cB}{4\pi ne} \\ &\cong C^2N/n\Omega_i. \\ \beta &\cong \frac{B^2\tau_i}{4\pi nM} \\ &\cong C^2\tau_i N/m, \end{aligned}$$

where $C = B/(4\pi NM)^{\frac{1}{2}}$ is again the characteristic Alfven speed. Hence we have the familiar result that

$$\beta/\alpha \cong \Omega_i\tau_i$$

in order of magnitude, showing that resistive diffusion is more important than the correction $\alpha\omega/C^2$ to b in the induction equation.

Note that in the same approximation

$$\eta \cong \frac{mc^2}{4\pi ne^2}\left(\frac{1}{\tau_e} + \frac{1}{\tau}\right),$$

so that

$$\frac{\alpha}{\eta} \cong \Omega_e\tau_e/(1+\tau_e/\tau),$$
$$\cong \Omega_e\tau_e,$$
$$\frac{\beta}{\eta} \cong \Omega_i\tau_i\,\Omega_e\tau_e,$$

in order of magnitude.

9. The Plane Polar Ionosphere

Before taking up some diffusive dynamical problems of ionospheric convection in the succeeding sections, it is useful to note briefly some elementary idealized circumstances in which an ionosphere without diffusion may be set in closed circulation by the Maxwell stress associated with magnetospheric convection. The response of the massive ionosphere is slow, of course, so it is also of interest to see how the stressed magnetic field arranges itself across a simple uniform ionospheric model when the motion of the ionosphere can be neglected. We cite a variety of simple solutions to (120) and (129).

9.1. Plane Model

Consider the polar ionosphere in the approximation that the mean field is vertical with a uniform strength B. Above a plane Earth the ionosphere is set in motion by the transport of magnetic flux bundles from the day side magnetosphere into the magnetotail as a consequence of reconnection at the magnetopause, etc. The net effect is to tilt the polar field away from the vertical, so that there is a Lorentz force exerted on the ionosphere by the tension in the tilted field, as sketched in Figure 7. In the simplest case, let the tilt of the field be independent of the horizontal coordinates x and y. The result of the tilt is a time dependent magnetic field in an accelerated ionosphere ($v_x, v_y \neq 0$). The horizontal magnetic perturbations B_x and B_y are small compared to the vertical component, of course, so that the quantities $b_x \equiv B_x/B$ and $b_y \equiv B_y/B$ are both small compared to one. Note that the horizontal uniformity yields $(\nabla \times \mathbf{b})_z = 0$ while the transverse component is $(\nabla \times \mathbf{b})_\perp = -e_x\partial b_y/\partial z + e_y \partial b_x/\partial z$. Equation (120) or (121) reduces to the linearized form

$$\frac{\partial b_x}{\partial t} \cong \frac{\partial}{\partial z}\left[v_x + (\eta+\beta)\frac{\partial b_x}{\partial z} + \alpha\frac{\partial b_y}{\partial z}\right], \qquad (133)$$

$$\frac{\partial b_y}{\partial t} \cong \frac{\partial}{\partial z}\left[v_y - \alpha\frac{\partial b_x}{\partial z} + (\eta+\beta)\frac{\partial b_y}{\partial z}\right], \qquad (134)$$

neglecting terms second order in b_x and b_y. The momentum equation (127) becomes

$$\frac{\partial v_x}{\partial t} = C^2\frac{\partial b_x}{\partial z} + \frac{\partial}{\partial z}\left(\mu\frac{\partial v_x}{\partial z}\right), \qquad (135)$$

$$\frac{\partial v_y}{\partial t} = C^2\frac{\partial b_y}{\partial z} + \frac{\partial}{\partial z}\left(\mu\frac{\partial v_y}{\partial z}\right) \qquad (136)$$

in the same approximation, noting that $(\mathbf{v}\cdot\nabla)\mathbf{v} = 0$. The quantity C is again the characteristic Alfven speed $B/(4\pi NM)^{\frac{1}{2}}$ and we have included a viscosity μ. The Lorentz force, given by (125), is

$$f_x = \frac{1}{4\pi}\frac{db_x}{dz}\;\; f_y = \frac{1}{4\pi}\frac{db_y}{dz}. \qquad (137)$$

Note, then, that the coefficients, α, β, η, μ, and C may be functions of z. The essential point is that (133)-(136) provide a complete set of four equations for time dependent v_x, v_y, b_x, and b_y.

If the electric current density $\mathbf{j}$ is needed for some purpose, it follows simply as

$$j_x = -\frac{cB}{4\pi}\frac{\partial b_y}{\partial z}\;\; j_y = +\frac{cB}{4\pi}\frac{\partial b_x}{\partial z}, \qquad (138)$$

The electric field follows from the linearized form of (117) with

$$E_x = \frac{B}{c}\left[-v_y + \alpha\frac{\partial b_x}{\partial z} - (\eta+\beta)\frac{\partial b_y}{\partial z}\right], \qquad (139)$$

$$E_y = \frac{B}{c}\left[+v_x + (\eta+\beta)\frac{\partial b_x}{\partial z} + \alpha\frac{\partial b_y}{\partial z}\right], \qquad (140)$$

from which it follows that the electric field $\mathbf{E}'$ in the frame of reference of the moving ionosphere is

$$E_x' = \frac{B}{c}\left[\alpha\frac{\partial b_x}{\partial z} - (\eta+\beta)\frac{\partial b_y}{\partial z}\right], \qquad (141)$$
$$= \frac{4\pi}{c^2}[(\eta+\beta)j_x + \alpha j_y],$$

$$E_y' = \frac{B}{c}\left[(\eta+\beta)\frac{\partial b_x}{\partial z} + \alpha\frac{\partial b_y}{\partial z}\right], \qquad (142)$$
$$= \frac{4\pi}{c^2}[-\alpha j_x + (\eta+\beta)j_y].$$

Solving these two expressions for j_x and j_y yields the familiar result

$$j_x = \frac{c^2}{4\pi}\frac{(\eta+\beta)E_x' - \alpha E_y'}{(\eta+\beta)^2+\alpha^2}, \qquad (143)$$

$$j_y = \frac{c^2}{4\pi}\frac{(\eta+\beta)E_x' + (\eta+\beta)E_y'}{(\eta+\beta)^2+\alpha^2}. \qquad (144)$$

It is well known, and obvious by inspection of 133)-(136), that there are Alfven waves, modified by the Hall resistivity α and dissipated by the Ohmic and Pedersen resistivity $\eta + \beta$. Time-dependent solutions $expi(\omega t + kz)$ in a uniform medium provide the usual dispersion relation

$$\left(\frac{\omega}{kC}\right)^2 - \left(\frac{\omega}{kC}\right)\frac{k}{C}[i(\eta+\beta)\pm\alpha] - 1 = 0, \qquad (145)$$

so that

$$\frac{\omega}{kC} = \pm\left\{1 + \frac{k^2}{4C^2}[\pm\alpha + i(\eta+\beta)]^2\right\}^{\frac{1}{2}} + \frac{k}{2C}[\pm\alpha + i(\eta+\beta)],$$

where the sign $\pm\alpha$ is independent of the $\pm$ in front of the braces, that is, there are four modes. The Pedersen and Ohmic resistive diffusion coefficients $(\beta + \eta)$ provide diffusion whereas the Hall resistive diffusion coefficient α couples b_x and b_y. With Hall resistivity alone $(\beta + \eta = 0)$ the result is

$$\frac{\omega}{kC} = \pm\left[1 + \left(\frac{\alpha k}{2C}\right)^2\right]^{\frac{1}{2}} \pm \frac{\alpha k}{2C},$$
$$\cong \pm 1 \pm \frac{\alpha k}{2C} + O^2\left(\frac{\alpha k}{2C}\right).$$

For $\alpha k/2C >> 1$, there are the respective fast and slow modes

$$\frac{\omega}{kC} = \pm\left(\frac{\alpha k}{C}\right)\left[1+\left(\frac{C}{\alpha k}\right)^2+\cdots\right],$$

$$\frac{\omega}{kC} = \pm\left(\frac{C}{\alpha k}\right)\left[1-\left(\frac{C}{\alpha k}\right)^2+\cdots\right].$$

On the other hand, in an ionosphere so dense that $\alpha << \eta+\beta$ the equations reduce to the usual resistive magnetohydrodynamic equations for a scalar field. There is only a single wave mode propagating in each direction along B.

9.2. Steady Acceleration of the Ionosphere

Suppose that the Maxwell stress $B^2 b_x/4\pi$ is maintained at a constant value at the top of the ionosphere while $b_y = 0$. The asymptotic form of the motion, after initial transients have died away, is

$$v_x = a_x t + U_x(z), \tag{146}$$

$$v_y = a_y t + U_y(z), \tag{147}$$

where a_x and a_y are constants. Presumably α, η, β, and the Alfven speed C are all functions of height z. Then if viscosity is neglected, (135) and (136) can be integrated immediately to give

$$b_x(z) = a_x \int_0^z \frac{dz'}{C(z')^2}, \tag{148}$$

$$b_y(z) = a_y \int_0^z \frac{dz'}{C(z')^2}, \tag{149}$$

satisfying the condition that the Maxwell stress across the lower surface ($z = 0$) of the ionosphere is zero. However, if there is no applied $b_y(L)$, it follows that $a_y = 0$. Hence $b_y(z) = 0$ throughout $0 < z < L$. The statement is then, that the Maxwell stress $B^2 b_x(z)/4\pi$ at the level z provides the acceleration a_x of the mass

$$\int_0^z dzNM = \frac{B^2}{4\pi}\int_0^z \frac{dz'}{C(z')^2}$$

up to that level, since there is no Maxwell stress across the lower boundary ($z = 0$) and we have omitted the viscous stress in the present illustrative example.

The induction equations (133) and (134) can be integrated over z, with the result

$$U_x(z) + (\eta+\beta)a_x/C^2 = D_1,$$

$$U_y(z) - \alpha a_x/C^2 = D_2,$$

where D_1 and D_2 are constants. Working in the fixed frame of reference in which $U_x(0) = U_y(0) = 0$, it follows that

$$U_x(z) = a_x\left[\left(\frac{\eta+\beta}{C^2}\right)_0 - \frac{\eta+\beta}{C^2}\right],$$

$$U_y(z) = a_x\left[\frac{\alpha}{C^2} - \left(\frac{\alpha}{C^2}\right)_0\right],$$

where the subscript zero denotes the value at $z = 0$. The final result is

$$v_x(z,t) = a_x\left[t + \left(\frac{\eta+\beta}{C^2}\right)_0 - \frac{\eta+\beta}{C^2}\right],$$

$$v_y(z,t) = a_x\left[\frac{\alpha}{C^2} - \left(\frac{\alpha}{C^2}\right)_0\right].$$

Note then that v_y is independent of time. It is nonvanishing only in so far as α/C^2 varies with z. The electric field follows from the (139) and (140) as the growing uniform field

$$E_x = a_x\frac{B}{c}\left(\frac{\alpha}{C^2}\right)_0 \quad E_y = a_x\frac{B}{c}\left[\left(\frac{\eta+\beta}{C^2}\right)_0 + t\right]. \tag{150}$$

The electric current is

$$j_x = 0,\ j_y = \frac{cBa_x}{4\pi C^2(z)}. \tag{151}$$

The total integrated current is just $(c/4\pi)Bb_x(L)$ from Ampere's law, of course. Note that applying the electric field E_y to the lumped ionospheric conductivity gives the wrong result because it is E'_y that drives the current rather than E_y. Note also that the problem of ionospheric acceleration by a specified Maxwell stress cannot be addressed by prescribing an applied electric field at the magnetopause because the electric field is not the driver and $\mathbf{E}$ is not known at the ionosphere until the dynamical equations have been solved. Using the $\mathbf{B}$, $\mathbf{v}$ paradigm to deduce the solution makes it possible to go back and restate the dynamics of the ionosphere in terms of the $\mathbf{E}$, $\mathbf{j}$ paradigm of course, but this ad hoc exercise serves no obvious purpose.

The ion and electron drift velocities can be computed from the linearized forms of (110) and (111) if they are needed. The electric drift (plasma) velocity in the tenuous magnetosphere ($z > L$) is

$$u_x = c\frac{E_y}{B},$$

$$= a_x\left[\left(\frac{\eta+\beta}{C^2}\right)_0 + t\right], \tag{152}$$

$$u_y = -c\frac{E_x}{B},$$

$$= -a_x\left(\frac{\alpha}{C^2}\right)_0. \tag{153}$$

The rate at which the Maxwell stress $e_x B^2 b_x(L)/4\pi$ does work per unit area on the ionosphere ($0 < z < L$) is

$$P = u_x B^2 b_x(L)/4\pi$$

$$= \frac{B^2 b_x(L) a_x}{4\pi}\left[\left(\frac{\eta+\beta}{C^2}\right)_0 + t\right] \tag{154}$$

The work goes into accelerating the ionosphere whose kinetic energy density is

$$T = \frac{1}{2}\rho(v_x^2 + v_y^2)$$

so that

$$\frac{dT}{dt} = \rho a_x^2\left[t + \left(\frac{\eta+\beta}{C^2}\right)_0 - \frac{\eta+\beta}{C^2}\right]. \tag{155}$$

The work also goes into resistive dissipation of density D with D/B^2 given by the two terms $4\pi\beta f^2$ and $\eta(\nabla \times \mathbf{b}^2)/4\pi$ on the right-hand side of (124). It follows that

$$D = a_x^2(\eta + \beta)\rho/C^2. \tag{156}$$

The total is

$$\int_0^L dz\left(\frac{dT}{dt} + D\right) = a_x \frac{B^2 b_x(L)}{4\pi}\left[\left(\frac{\eta+\beta}{C^2}\right)_0 + t\right] \tag{157}$$

which is precisely equal to the energy input rate P of course. The Lorentz force **f** is given by (137).

9.3. Asymptotic Field Neglecting Ionospheric Wind

In the idealized case of an ionosphere that is so massive that v_x and v_y are negligible compared to the rate of diffusion of the field put $v_x = v_y = 0$ in (133) and (134). After a time of the order of $L^2/(\eta + \beta)$ the field relaxes asymptotically to a final steady state $(\partial/\partial t = 0)$ in which (133) and (134) can be integrated once to yield

$$(\eta+\beta)\frac{db_x}{dz} + \alpha\frac{db_y}{dz} = C_1, \tag{158}$$

$$-\alpha\frac{db_x}{dz} + (\eta+\beta)\frac{db_y}{dz} = C_2, \tag{159}$$

where C_1 and C_2 are constants. These two equations can be solved for db_x/dz and db_y/dz and integrated again yielding

$$b_x(z) = C_1\Sigma_1(z) - C_2\Sigma_2(z), \tag{160}$$

$$b_y(z) = C_1\Sigma_2(z) + C_2\,\Sigma_1(z), \tag{161}$$

where the integrals Σ_1 and Σ_2 are defined as

$$\Sigma_1(z) \equiv \int_0^z \frac{dz(\eta+\beta)}{(\eta+\beta)^2+\alpha^2}, \tag{162}$$

$$\Sigma_2(z) \equiv \int_0^z \frac{dz\,\alpha}{(\eta+\beta)^2+\alpha^2}, \tag{163}$$

and represent the height integrated conductivities. With $b_x(L)$ given and $b_y(L) = 0$ it follows that

$$C_1 = b_x(L)\Sigma_1(L)/[\Sigma_1(L)^2 + \Sigma_2(L)^2],$$

$$C_2 = -b_x(L)\Sigma_2(L)/[\Sigma_1(L)^2 + \Sigma_2(L)^2],$$

so that

$$b_x(z) = b_x(L)\frac{\Sigma_1(L)\Sigma_1(z) + \Sigma_2(L)\Sigma_2(z)}{\Sigma_1(L)^2 + \Sigma_2(L)^2}, \tag{164}$$

$$b_y(z) = b_x(L)\frac{\Sigma_1(L)\Sigma_2(z) - \Sigma_2(L)\Sigma_1(z)}{\Sigma_1(L)^2 + \Sigma_2(L)^2}. \tag{165}$$

In the idealized circumstance of a vertically uniform ionosphere it follows that

$$b_x(z) = b_x(L)z/L,$$

$$b_y(z) = 0.$$

If for some reason we need the electric current density it follows from Ampere's law (138) that

$$j_x(z) = +$$

$$\frac{cB}{4\pi}b_x(L)\frac{(\eta+\beta)\Sigma_2(L) - \alpha\Sigma_1(L)}{[(\eta+\beta)^2+\alpha^2][\Sigma_1(L)^2+\Sigma_2(L)^2]}, \tag{166}$$

$$j_y(z) =$$

$$\frac{cB}{4\pi}b_x(L)\frac{(\eta+\beta)\Sigma_1(L) + \alpha\Sigma_2(L)}{[(\eta+\beta)^2+\alpha^2][\Sigma_1(L)^2+\Sigma_2(L)^2]}. \tag{167}$$

The electric field follows from (139) and (140) as

$$E_x = \frac{Bb_x(L)}{c}\,\frac{\Sigma_2(L)}{[\Sigma_1^2(L) + \Sigma_2^2(L)]}, \tag{168}$$

$$E_y = \frac{Bb_x(L)}{c}\,\frac{\Sigma_1(L)}{\Sigma_1(L)^2 + \Sigma_2(L)^2}. \tag{169}$$

One notes that the total integrated Lorentz force is equal to the total Maxwell stress exerted on the ionosphere which follows from (137) as

$$B^2\int_0^L dz f_x = \frac{B^2}{4\pi}b_x(L).$$

The rate at which the Maxwell stress does work on the ionosphere is just the Maxwell stress multiplied by the speed of the magnetosphere cE_y/B above the top of the ionosphere. The power input is then

$$P = cb_x(L)BE_y/4\pi,$$

$$= \frac{B^2 b_x(L)^2\Sigma_1(L)}{4\pi[\Sigma_1(L)^2 + \Sigma_2(L)^2]}. \tag{170}$$

Turning to the energy equation (124) the two dissipative terms on the right side are

$$4\pi\beta f^2 + \eta\Big(\nabla\times\mathbf{b}\Big)^2/4\pi$$

$$= \frac{\beta}{4\pi}\left[\left(\frac{db_x}{dz}\right)^2 + \left(\frac{db_x}{dz}\right)^2\right] + \frac{4\pi\eta}{c^2}(j_x^2 + j_y^2),$$

$$= \frac{b_x(L)^2(\eta+\beta)}{4\pi[\Sigma_1^2(L)+\Sigma_2^2(L)][(\eta+\beta)^2+\alpha^2]}. \tag{171}$$

Multiply by B^2 to dimensionalize the result to ergs/cm^3 sec and integrate over z from 0 to L. The result is just P of course.

9.4. Field Relaxation Neglecting Ionospheric Wind

The next problem is the time-dependent relaxation of the magnetic field in an idealized massive ionosphere whose bulk motion can be neglected. It is sufficient for this preliminary study to treat the idealized case in which the ionosphere is a uniform slab of thickness L with α, β, and η uniform across L.

The calculation begins with (133) and (134) where v_x and v_y are put equal to zero. The equations can be written

$$\left[\frac{\partial}{\partial t} - (\eta+\beta)\frac{\partial^2}{\partial z^2}\right]b_x = +\alpha\frac{\partial^2 b_y}{\partial z^2}; \tag{172}$$

$$\left[\frac{\partial}{\partial t} - (\eta + \beta)\frac{\partial^2}{\partial z^2}\right] b_y = -\alpha \frac{\partial^2 b_x}{\partial z^2}, \qquad (173)$$

in a uniform ionosphere. Hence both b_x and b_y satisfy the equation

$$\left[\frac{\partial}{\partial t} - (\eta + \beta)\frac{\partial^2}{\partial z^2}\right]^2 \psi + \alpha^2 \frac{\partial^4 \psi}{\partial z^4} = 0 \qquad (174)$$

The general solution of these equations is expressible in terms of an infinite sum over the separable solutions. We consider some special cases to illustrate the separate contributions from α and $\beta + \eta$ before treating the general case. Consider first the idealized situation in which the ionosphere is so dense that $\Omega_i \tau_i$, $\Omega_e \tau_e << 1$, where Ω_i and Ω_e are the ion and electron cyclotron frequencies eB/Mc and eB/mc, respectively. (Recall that the ion and electron mean free paths for collision with neutral atoms are comparable in order of magnitude). Hence $\tau_i/\tau_e \sim O(M^{\frac{1}{2}}/m^{\frac{1}{2}})$ and $\Omega_i \tau_i$ is small compared to $\Omega_e \tau_e$ by $O(m^{\frac{1}{2}}/M^{\frac{1}{2}})$. Neglecting $m/\tau_e = (eB/c)/\Omega_e \tau_e$ compared to $M/\tau_i = (eB/c)/\Omega_i \tau_i$ in (114)-(116) yields $\beta/\alpha \cong \Omega_i \tau_i$, $\alpha/\eta \cong \Omega_e \tau_e$.) Then $\alpha, \beta << \eta$ and (174) reduces to

$$[\frac{\partial}{\partial t} - \eta \frac{\partial^2}{\partial z^2}]\psi = 0. \qquad (175)$$

Note again that b_x and b_y vanish at the top ($z = 0$) of the massive nonconducting atmosphere. The magnetosphere of collisionless plasma ($\eta + \beta = 0$) lies above $z = L$ and has negligible mass. Thus the magnetic field in $z > L$ can change quickly with time as a consequence of the transport of magnetic field at the distant magnetopause. ($z = \Lambda >> L$). So it is interesting to see the consequences of various time-dependent forms of the tilt $b_x(L)$ of the field applied to the top ($z = L$) of the ionosphere.

Suppose then that $b_y(L) = 0$ for all t while $b_x(L) = 0$ for $t < 0$, jumping suddenly to the fixed value b_0 at time $t = 0$. Maintaining $b_x = b_0$ at $z = L$ for $t > 0$ requires horizontal displacement in the x direction at the magnetopause $z = \Lambda$. The field for $t > 0$ is readily shown to be

$$b_x(z,t)$$

$$= b_0\left[\frac{z}{L} + \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n sin\frac{n\pi z}{L} exp\left(-\frac{n^2\pi^2\eta t}{L^2}\right)\right] \qquad (176)$$

and for $\eta t >> L$,

$$b_x(z,t) \sim b_0\left[\frac{z}{L} - \frac{2}{\pi} sin\frac{\pi z}{L} exp\left(-\frac{\pi^2 \eta t}{L^2}\right) + \cdots\right] \qquad (177)$$

as the field relaxes to the asymptotic static form $b_0 z/L$.

Alternatively, suppose that the tilt $b_x(L)$ of the field is switched on suddenly at time $t = 0$ in $z > L$ but thereafter relaxes back to zero. A simple case is given by the applied transverse component $b_x(L,t)$ in the form

$$b_x(L,t) = b_0\, erf[L/(\eta t)^{\frac{1}{2}}] \qquad (178)$$

for $t > 0$ where erf denotes the error function. Then since $erf(\infty) = 1$ the tilt b_x jumps from 0 to b_0 at time $t = 0$ and subsequently returns asymptotically to zero as $t^{-1/2}$ with

$$b_x(L,t) \sim b_0\left(\frac{4L^2}{\pi\eta t}\right)^{\frac{1}{2}}\left[1 - \frac{L^2}{3\eta t} + \cdots\right] \qquad (179)$$

for large t. The resulting field or tilt b_x throughout $0 < z < n$ is

$$b_x(z,t)$$

$$= b_0\left[erf[(L+z)/(4\eta t)^{\frac{1}{2}}] - erf[(L-z)/(4\eta t)^{\frac{1}{2}}]\right] \qquad (180)$$

$$\sim \frac{2b_0 z}{(\pi\eta t)^{\frac{1}{2}}}\left(1 - \frac{3L^2 + z^2}{12\eta t} + \cdots\right), \qquad (181)$$

declining asymptotically as $z/t^{\frac{1}{2}}$ in response to the declining tilt applied at $z = L$ but varying linearly with z up across the ionosphere. The electric current is given by $(cB/4\pi)\partial b_x/\partial z$ if it is needed. The Lorentz force exerted by the Maxwell stress $Bb_x/4\pi$ is proportional to $\partial b_x/\partial z$ so that it is uniform across the height of the ionosphere. Other examples are given in Appendix C.

In the opposite extreme ($\alpha >> \eta + \beta$) for a tenuous ionosphere the field equation (174) approximates to

$$\frac{\partial^2 \psi}{\partial t^2} + \alpha^2 \frac{\partial^4 \psi}{\partial z^4} = 0. \qquad (182)$$

This equation has the separable solutions $expi(\omega t + kx)$ with $\omega = \pm\alpha k^2$ representing waves with a phase velocity $v_\psi = \omega/k = \pm\alpha k$ and a group velocity $d\omega/dk = \pm 2\alpha k$. There are similarity solutions of the form $t^a \Psi(\xi)$ where $\xi = z/t^{\frac{1}{2}}$ and Ψ satisfies

$$\alpha^2 \Psi^{IV} + \tfrac{1}{4}\xi^2 \Psi^{II} + (\tfrac{3}{4} - a)\xi\Psi^{I} + a(a-1)\Psi = 0. \qquad (183)$$

where the roman numeral superscripts indicate derivatives.

Setting $a = -\frac{1}{2}$ yields the two solutions

$$\Psi = sin\frac{z^2}{4\alpha t},\ cos\frac{z^2}{4\alpha t},$$

the first of which satisfies the boundary condition that $b_x = b_y = 0$ at $z = 0$. Therefore we have from (172) and (173) (with $\eta = \beta = 0$) the solution

$$b_x = +t^{-1/2} sin\frac{z^2}{4\alpha t},\ b_y = +t^{-1/2} cos\frac{z^2}{4\alpha t},$$

and the complementary solution

$$b_x = +t^{-1/2} cos\frac{z^2}{4\alpha t},\ b_y = -t^{-1/2} sin\frac{z^2}{4\alpha t}.$$

But only one of b_x, b_y can be made to vanish at $z = 0$ in this case. Polynomial solutions in ξ are easily constructed for (182) and the first few orders are exhibited in Appendix D. The polynomials that contain only the first power of t vanish at $z = 0$ satisfying the boundary conditions on b_x and b_y. Polynomials of second or higher order in t do not vanish at $z = 0$ and so are unsuitable in the present context. It is obvious by substitution into (172) and (173) that one solution is

$$b_x(z,t) = \alpha z t,\ b_y(z,t) = \tfrac{1}{6}z^3 \qquad (184)$$

for which $b_x = b_y = 0$ at $z = 0$. The complementary solution is

$$b_x(z,t) = \tfrac{1}{6}z^3, b_y(z,t) = -\alpha z t \qquad (185)$$

The rotation of the vector $(b_x,\ b_y)$ over height z is a consequence of the Hall effect, of course.

The general case, retaining both α and $\eta+\beta$, employs the full equation (174). The separable solution $expi(\omega t + kz)$ provides the dispersion relation $\omega = k^2[\pm\alpha + i(\eta+\beta)]$. The similarity solutions $t^a G(\xi)$ are described by the fourth order differential equation

$$\begin{gathered}[(\eta+\beta)^2+\alpha^2]G^{IV} \\ +(\eta+\beta)\xi G^{III} + \left[\tfrac{1}{4}\xi^2 - 2(a-1)(\eta+\beta)\right]G^{II} \qquad (186) \\ +(\tfrac{3}{4}-a)\xi G' + a(a-1)G = o\end{gathered}$$

which we note is invariant under the transformation $\xi \rightarrow -\xi$. There is again a variety of polynomial solutions. For instance if $a = \frac{3}{2}$, the solution $G = \xi$ leads to

$$b_x = +\alpha z t,\ b_y = z\left[\tfrac{1}{6}z^2 + (\eta+\beta)t\right] \qquad (187)$$

and the complementary solution

$$b_x = z\left[\tfrac{1}{6}z^2 + (\eta+\beta)t\right],\ b_y = -\alpha z t \qquad (188)$$

of (172) and (173). These solutions satisfy the boundary condition that $b_x = b_y = 0$ at $z = \xi = 0$. They exhibit a Maxwell stress (tilt of the field) growing linearly with time starting from $b_x = 0,\ b_y = \frac{1}{6}z^3$ or $b_x = \frac{1}{6}z^3,\ b_y = 0$ at time $t = 0$. Additional polynomial solutions are indicated in Appendix D. The purpose of the foregoing array of mathematical solutions has been to illustrate the application of the **B**,**v** paradigm to motion of a partially ionized medium. The next section takes up the combined motion and diffusion, going to a steady state to minimize the computation. As the reader can readily demonstrate, approaching the problem with the **E**, **j** paradigm is problematical unless the solution from the **B**, **v** paradigm is available as a guide.

10. Steady Motion of the Polar Ionosphere

If the steady Maxwell stress $B^2 b(L)/4\pi$ of the tilted magnetic field is applied at the upper surface $z = L$ of the ionosphere for a sufficiently long time, the massive idealized ionosphere eventually achieves a steady motion $v_x(z),\ v_y(z)$ limited only by the viscous drag on the lower boundary $z = 0$. Viscosity may drag the nonconducting atmosphere in $z < 0$ along with the ionosphere to some small degree so we work again in the frame of reference of the motion at $z = 0$, putting $v_x = v_y = b_x = b_y = 0$ at $z = 0$. Under stationary conditions $(\partial/\partial t = 0)$ and the idealization that α, β, η, μ, and C are independent of z, (133) - (136) can be integrated over z to give

$$v_x + (\eta+\beta)\frac{db_x}{dz} + \alpha\frac{db_y}{dz} = V_1 \qquad (189)$$

$$v_y - \alpha\frac{db_x}{dz} + (\eta+\beta)\frac{db_y}{dz} = V_2 \qquad (190)$$

$$C^2 b_x + \nu\frac{dv_x}{dz} = V_3 \qquad (191)$$

$$C^2 b_y + \nu\frac{dv_y}{dz} = V_4 \qquad (192)$$

where ν is the kinematic viscosity μ/ρ and where the four V_i are constants, to be evaluated by applying the boundary conditions.

Now $b_y(L) = 0$ and, since there is no viscous stress in $z > L$, it follows that $dv_x/dz = dv_y/dz = 0$, as z increases to L. Hence (191) and (192) yield

$$V_3 = C^2 b_x(L), V_4 = 0. \qquad (193)$$

Use (189) and (190) to eliminate v_x and v_y from (191) and (192), obtaining

$$b_x - \nu\frac{(\eta+\beta)}{C^2}\frac{d^2b_x}{dz^2} - \frac{\alpha\nu}{C^2}\frac{d^2b_y}{dx^2} = \frac{V_3}{C^2}, \qquad (194)$$

$$\frac{\alpha\nu}{C^2}\frac{d^2b_x}{dz^2} + b_y - \frac{\nu(\eta+\beta)}{C^2}\frac{d^2b_y}{dz^2} = \frac{V_4}{C^2}. \qquad (195)$$

Then let

$$b_x = \frac{V_3}{C^2} + Q_1 expqz, \qquad (196)$$

$$b_y = \frac{V_4}{C^2} + Q_2 expqz, \qquad (197)$$

where Q_1 and Q_2 are constants. The result is

$$[C^2/\nu - (\eta+\beta)q^2]Q_1 - \alpha q^2 Q_2 = 0, \qquad (198)$$

$$\alpha q^2 Q_1 + [C^2/\nu - (\eta+\beta)q^2]Q_2 = 0. \qquad (199)$$

Setting the determinant equal to zero yields the two possibilities

$$q_{1,2}^2 = \frac{C^2(\eta+\beta\pm i\alpha)}{\nu[(\eta+\beta)^2+\alpha^2]}, q_{1,2}^{-2} = \frac{\nu}{C^2}(\eta+\beta\mp i\alpha), \qquad (200)$$

so that there are four roots $\pm q_1$ and $\pm q_2$, with $q_2 = q_1^*$ and

$$\begin{aligned} q_1 &= \frac{C}{\nu^{\frac{1}{2}}\,[\eta+\beta-i\alpha]^{\frac{1}{2}}}, \\ &= \frac{C(cos\theta + isin\theta)}{\nu^{\frac{1}{2}}[(\eta+\beta)^2+\alpha^2]^{\frac{1}{4}}} \end{aligned} \qquad (201)$$

with

$$2^{\frac{1}{2}}cos\theta = \left\{1 + \frac{\eta+\beta}{[(\eta+\beta)^2+\alpha^2]^{\frac{1}{2}}}\right\}^{\frac{1}{2}}. \qquad (202)$$

$$2^{\frac{1}{2}}sin\theta = \left\{1 - \frac{\eta+\beta}{[(\eta+\beta)^2+\alpha^2]^{\frac{1}{2}}}\right\}^{\frac{1}{2}}. \qquad (203)$$

For $q = \pm q$, we have $Q_2 = -iQ_1$, while for $q = \pm q_2, Q_2 = +iQ_1$.

The complete solution has the form

$$\begin{gathered} b_x(z) = \frac{V_3}{C^2} + K_1 expq_1 z \\ +K_2 exp(-q_1 z) + K_3 expq_2 z + K_4 exp(-q_2 z), \qquad (204)\end{gathered}$$

$$b_y(z) = \frac{V_4}{C^2} - iK_1 exp q_1 z -$$

$$iK_2 exp(-q_1 z) + iK_3 exp q_2 z + iK_4 exp(-q_2 z), \quad (205)$$

where the K_i are arbitrary constants. The velocity (v_x, v_y) follows from (189) and (190)

The boundary conditions that $v_x = v_y = b_x = b_y = 0$ at $z = 0$ require the four relations

$$\nu V_1/C^2 - (K_1 - K_2)/q_1 - (K_3 - K_4)/q_2 = 0, \quad (206)$$

$$\nu V_2/C^2 + i(K_1 - K_2)/q_1 - i(K_3 - K_4)/q_2 = 0, \quad (207)$$

$$V_3/C^2 + K_1 + K_2 + K_3 + K_4 = 0, \quad (208)$$

$$V_4/C^2 - i(K_1 + K_2) + i(K_3 + K_4) = 0, \quad (209)$$

respectively, upon making use of the fact that

$$\eta + \beta \mp i\alpha = C^2/\nu q_{2,1}^2.$$

Now the boundary conditions at $z = L$ are $b_y(L) = 0$ with $dv_x/dz = dv_y/dz = 0$, yielding (193). There is no viscous stress in $z > L$ and no acceleration in $z < L$, so $d^2v_x/dz^2 = d^2v_y/dz^2 = 0$ as z approaches L, requiring

$$K_1 exp q_1 L + K_2 exp(-q_1 L)$$

$$+K_3 exp q_2 L + K_4 exp(-q_2 L) = 0, \quad (210)$$

$$K_1 exp q_1 L + K_2 exp(-q_1 L)$$

$$-K_3 exp q_2 L - K_4 exp(-q_2 L) = 0. \quad (211)$$

It follows from (211) and (212) that

$$K_2 = -K_1 exp 2q_1 L, K_4 = -K_3 exp 2q_2 L \quad (212)$$

These two relations combined with (208) and (209) yield

$$K_1 = -\frac{V_3 + iV_4}{2C^2(1 - exp 2q_1 L)},$$

$$= -\frac{b_x(L)}{2(1 - exp 2q_1 L)}, \quad (213)$$

$$K_3 = -\frac{V_3 - iV_4}{2C^2(1 - exp 2q_2 L)},$$

$$= -\frac{b_x(L)}{2(1 - exp 2q_2 L)}, \quad (214)$$

upon using (193) for V_3 and V_4. The constants V_1 and V_2 follow from (206) and (207) as

$$V_1 = \frac{C^2 b_x(L)}{2\nu}\left(\frac{coth q_1 L}{q_1} + \frac{coth q_2 L}{q_2}\right), \quad (215)$$

$$V_2 = \frac{iC^2 b_x(L)}{2\nu}\left(-\frac{coth q_1 L}{q_1} + \frac{coth q_2 L}{q_2}\right), \quad (216)$$

Equations (204) and (205) can now be written

$$b_x(z)$$

$$= b_x(L)\left[1 - \frac{sinh q_1(L-z)}{2 sinh q_1 L} - \frac{sinh q_2(L-z)}{2 sinh q_2 L}\right], \quad (217)$$

$$b_y(z)$$

$$= \frac{1}{2} i b_x(L)\left[\frac{sinh q_1(L-z)}{sinh q_1 L} - \frac{sinh q_2(L-z)}{sinh q_2 L}\right]. \quad (218)$$

The velocity components follow from (189) and (190) with

$$v_x(z) = \frac{C^2 b_x(L)}{2\nu}\left[\frac{cosh q_1 L - cosh q_1(L-z)}{q_1 sinh q_1 L}\right.$$

$$\left. + \frac{cosh q_2 L - cosh q_2(L-z)}{q_2 sinh q_2 L}\right], \quad (219)$$

$$v_2(z) = \frac{iC^2 b_x(L)}{2\nu}\left[-\frac{cosh q_1 L - cosh q_1(L-z)}{q_1 sinh q_1 L}\right.$$

$$\left. + \frac{cosh q_2 L - cosh q_2(L-z)}{q_2 sinh q_2 L}\right]. \quad (220)$$

The Lorentz force $(\nabla \times \mathbf{b}) \times b/4\pi$ exerted on the fluid is

$$f_x = \frac{1}{4\pi}\frac{db_x}{dz},$$

$$= \frac{b_x(L)}{8\pi}\left[\frac{q_1 cosh q_1(L-z)}{sinh q_1 L} + \frac{q_2 cosh q_2(L-z)}{sinh q_2 L}\right], \quad (221)$$

$$f_y = \frac{1}{4\pi}\frac{db_y}{dz}, = \frac{ib_x(L)}{8\pi}$$

$$\left[-\frac{q_1 cosh q_1(L-z)}{sinh q_1 L} + \frac{q_2 cosh q_2(L-z)}{sinh q_2 L}\right]. \quad (222)$$

For the record the current density is

$$j_x = -\frac{cB}{4\pi}\frac{db_y}{dz},$$

$$= -cBf_y, \quad (223)$$

$$j_y = +\frac{cB}{4\pi}\frac{db_x}{dz},$$

$$= +cBf_x. \quad (224)$$

The electric field follows from (113) and the (189) and (190) as

$$E_x = B\left[-\frac{v_y}{c} - (\eta + \beta)\frac{db_y}{dz} + \alpha\frac{db_x}{dz}\right],$$

$$= -V_2 B/c, \quad (225)$$

$$E_y = B\left[\frac{v_x}{c} + (\eta + \beta)\frac{db_x}{dz} + \alpha\frac{db_y}{dz}\right],$$

$$= +V_1 B/c, \quad (226)$$

where V_1 and V_2 are given by(215) and (216).

The motion of the magnetosphere ($z > L$) is given by $c\mathbf{E} \times \mathbf{B}/B^2$, or

$$v_x = +cE_y/B, \quad = V_1, \tag{227}$$

$$v_y = -cE_x/B, \quad = +V_2. \tag{228}$$

Note from (219) and (220) that the motion of the neutral gas at the upper boundary of the ionosphere ($z = L$) is

$$v_x(L) = V_1 - \frac{C^2 b_x(L)}{2\nu}\left(\frac{1}{q_1 sinh q_1 L} + \frac{1}{q_2 sinh q_2 L}\right), \tag{229}$$

$$v_y(L) = V_2 + \frac{iC^2 b_x(L)}{2\nu}\left(\frac{1}{q_1 sinh q_1 L} - \frac{1}{q_2 sinh q_2 L}\right). \tag{230}$$

Comparing (229) and (230) with (227) and (228) it is evident that the magnetospheric plasma does not move with the same velocity as the neutral gas at the top of the ionosphere. The motion of the magnetospheric plasma is different from the drift of the ions and the electrons, too. The magnetospheric plasma moves with the electric drift velocity, whereas the ionospheric constituents all drift slowly in various ways relative to that frame of reference as a consequence of the collisional forces and the viscous drag.

The driving force $B^2 b_x(L)/4\pi$ is in the x direction. The y component of the ionospheric velocity v_y is nonvanishing as a consequence of the viscosity and collisional drag through the Hall effect, of course. This follows formally upon noting from (201) - (203) that putting $\alpha = 0$ leads to $q_1 = q_2$. Equation (216) gives $V_2 = 0$ so that (225) yields $E_x = 0$ and (220) yields $v_y(z) = 0$. Newton's third law requires that the viscous drag across the lower surface ($z = 0$) of the ionosphere is also in the x direction and equal to the driving force at the upper surface of the magnetosphere. Differentiation of (220) establishes that dv_y/dz vanishes at $z = 0$, while differentiation of (219) yields the viscous stress $\mu dv_x/dz$ at $z = 0$ as being precisely equal to the applied Maxwell stress

$$\mu\frac{dv_x}{dz} = \frac{B^2 b_x(L)}{4\pi}.$$

To examine the solutions in more detail, note that the ion and electron drift velocities follow from (110), (111), (189), and (190). It is convenient to write

$$\alpha_1 \equiv \frac{cBM/\tau_i}{4\pi neQ}, \alpha_2 \equiv \frac{cBm/\tau_e}{4\pi neQ}, \tag{231}$$

so that

$$\alpha = \alpha_1 - \alpha_2,\ cB/4\pi ne = \alpha_1 + \alpha_2. \tag{232}$$

Then, using primes to denote differentiation with respect to z, the ion drift velocity is

$$\begin{aligned} w_x &= v_x + \beta b_x' - \alpha_2 b_y', \\ &= V_1 - \eta b_x' - \alpha_1 b_y', \\ &= V_1 - \frac{1}{2}b_x(l)\left[\frac{(\eta - i\alpha_1)q_1 cosh q_1(L-z)}{sinh q_1 L} + \frac{(\eta + i\alpha_1)q_2 cosh q_2(L-z)}{sinh q_2 L}\right], \end{aligned} \tag{233}$$

$$\begin{aligned} w_y &= v_y + \beta b_y' + \alpha_2 b_x', \\ &= V_2 + \alpha_1 b_x' - \eta b_y', \\ &= V_2 + \frac{i}{2}b_x(l)\left[\frac{(\eta - i\alpha_1)q_1 cosh q_1(L-z)}{sinh q_1 L} - \frac{(\eta + i\alpha_1)q_2 cosh q_2(L-z)}{sinh q_2 L}\right]. \end{aligned} \tag{234}$$

The electron drift velocity is

$$\begin{aligned} u_x &= v_x + \beta b_x' + \alpha_2 b_y', \\ &= V_1 - \eta b_x' + \alpha_2 b_y', \\ &= V_1 - \frac{1}{2}b_x(L)\left[\frac{(\eta + i\alpha_2)q_1 cosh q_1(L-z)}{sinh q_1 L} + \frac{(\eta - i\alpha_2)q_2 cosh q_2(L-z)}{sinh q_2 L}\right]. \end{aligned} \tag{235}$$

$$\begin{aligned} u_y &= v_y + \beta b_x' - \alpha_1 b_y', \\ &= V_2 - \alpha_2 b_x' - \eta b_y', \\ &= V_2 + \frac{i}{2}b_x(l)\left[\frac{(\eta + i\alpha_2)q_1 cosh q_1(L-z)}{sinh q_1 L} - \frac{(\eta - i\alpha_2 q_2 cosh q_2(L-z)}{sinh q_2 L}\right]. \end{aligned} \tag{236}$$

With these results it is easy to show from (105) or (106) that

$$j_x = -\frac{icB}{8\pi}b_x(L) \times \left[\frac{q_1 cosh q_1(L-z)}{sinh q_1 L} - \frac{q_2 cosh q_2(L-z)}{sinh q_2 L}\right], \tag{237}$$

$$j_y = +\frac{cB}{8\pi}b_x(l) \times \left[\frac{q_1 cosh q_1(L-z)}{sinh q_1 L} + \frac{q_2 cosh q_2(L-z)}{sinh q_2 L}\right]. \tag{238}$$

Since $q_2 = q_1{}^*$, it is obvious that $w_x, w_y, u_x, u_y, j_x, j_y$ are all real quantities.

When $q_{1,2}L << 1$, it is readily shown from (217) and (218) that

$$b_x(z) \cong b_x(L)\frac{z}{L} \times \left[1 - \frac{C^2(\eta+\beta)(L-z)(L-\frac{1}{2}z)}{3\nu[(\eta+\beta)^2+\alpha^2]} + O^4(q_{1,2}L)\right], \tag{239}$$

$$b'_z(z) = \frac{b_x(L)}{L}$$

$$\times\left[1-\frac{C^2(\eta+\beta)(L^2-3Lz+\frac{3}{2}z^2)}{3\nu[(\eta+\beta)^2+\alpha^2]}+O^4(q_{1,2}L)\right], \quad (240)$$

$$b_y(z) \cong -b_x(L)$$

$$\times\frac{C^2\alpha z(L-z)(L-\frac{1}{2}z)}{L\nu[(\eta+\beta)^2+\alpha^2]}[1+O^2(q_{1,2}L)], \quad (241)$$

$$b'_y(z) \cong$$

$$-\frac{b_x(L)}{L}\frac{C^2\alpha(L^2-3Lz+\frac{3}{2}z^2)}{3\nu[(\eta+\beta)^2+\alpha^2]}[1+O^2(q_{1,2}L)]. \quad (242)$$

It follows from (215) and (217) that

$$V_1 \cong \frac{b_x(L)}{L}(\eta+\beta)\left[1+\frac{C^2L^2}{3\nu(\eta+\beta)}+O^4(q_{1,2}L)\right], \quad (243)$$

$$V_2 = -\frac{b_x(L)}{L}\alpha[1+O^4(q_{1,2}L)]. \quad (244)$$

The electric current densities j_x and j_y are proportional to $-b'_y$ and $+b'_x$, respectively, through Ampere's law, of course.

The energy budget of the ionosphere is easily elaborated. The energy flux is given by the z component of the Poynting vector evaluated at $z = L$, that is, by the Maxwell stress multplied by the electric drift velocity, so that

$$P_z(z) = \frac{CB}{4\pi}(E_x b_y - E_y b_x),$$

$$= -\frac{B^2}{4\pi}(V_1 b_x + V_2 b_y). \quad (245)$$

At $z = L$ this reduces to

$$P_z(L) = -\frac{B^2}{4\pi}V_1 b_x(L). \quad (246)$$

It is evident from (227) that this is just the x-component of the velocity of the magnetosphere multiplied by the applied Maxwell stress [*Zhu*, 1994a]. It is, of course, also equal to $\mathbf{E}\cdot\mathbf{J}$, where $\mathbf{J}$ is the total integrated current across the height of the ionosphere.

The energy is withdrawn from the Poynting vector in the ionosphere at the rate given by the right-hand side of (124). Noting that the divergence term is identically zero because $\partial/\partial x = \partial/\partial y = 0$, the rate D ergs / cm^{-3} s is

$$D = B^2[\mathbf{f}\cdot\mathbf{v} + 4\pi\beta f^2 + \eta(\Delta\times\mathbf{b})^2/4\pi], \quad (247)$$

$$= \frac{B^2}{4\pi}[b'_x v_x + b'_y v_y + (\beta+\eta)(b'^2_x + b'^2_y)],$$

since $f_i = b'_i/4\pi$. Eliminating v_x and v_y through (189) and (190), it follows that

$$D = \frac{B^2}{4\pi}(V_1 b'_x + V_2 B'_y). \quad (248)$$

Comparing this result with (245) for $P_z(z)$, it is obvious that $dP_z/dz = -D$. The Poynting vector falls to zero at the lower boundary ($z = 0$) of the ionosphere where $b_x = b_y = 0$, of course.

The energy delivered into the ionosphere by the Maxwell stress, that is, by the Poynting vector, is dissipated into heat via viscosity and via the combined Ohmic and Pedersen resistivities. The term $B^2\mathbf{f}\cdot\mathbf{v}$ on the right-hand side of (247) represents the rate at which the Lorentz force $B^2\mathbf{f}$ does work on the neutral atmosphere in opposition to the viscosity. It is readily shown that

$$B^2\mathbf{f}\cdot\mathbf{v} = \frac{B^2}{4\pi}[V_1 b'_x + V_2 b'_y - (\eta+\beta)(b'^2_x + b'^2_y)], \quad (249)$$

wherein b'_x and b'_y follow from (217) and (218) while the remaining terms on the right-hand side of (247) reduce to the resistive dissipation

$$B^2[4\pi\beta f^2+\eta(\Delta\times\mathbf{b})^2/4\pi] = \frac{B^2}{4\pi}(\eta+\beta)(b'^2_x+b'^2_y). \quad (250)$$

The sum reduces, then, to the total given by (248). The viscous dissipation rate (ergs / cm^{-3} s) is

$$\rho\nu(v'^2_x + v'^2_y) =$$

$$\frac{B^2}{4\pi}\frac{C^2}{\nu}b_x(l)^2\frac{sinhq_1(L-z)sinhq_2(L-z)}{sinhq_1L\ sinhq_2L}, \quad (251)$$

so that the total viscous dissipation across $(0, L)$ is

$$\int_0^L \rho\nu(v'^2_x + v'^2_y) = -\frac{iB^2}{8\pi}\times$$

$$\frac{[(\eta+\beta)^2+\alpha^2]}{\alpha}b_x(L)^2[q_1cothq_1L - q_2cothq_2L]. \quad (252)$$

The resistive dissipation is

$$\frac{B^2}{4\pi}(\eta+\beta)(b'^2_x + b'^2_y) = \frac{B^2}{4\pi}$$

$$\times\frac{(\eta+\beta)q_1q_2b_x(L)^2}{sinhq_1L\ sinhq_2L}coshq_1(L-z)coshq_2(L-z), \quad (253)$$

and the integrated total is

$$\frac{B^2}{4\pi}(\eta+\beta)\int_0^L dz(b'^2_x + b'^2_y)$$

$$= \frac{iB^2}{8\pi}\frac{(\eta+\beta)C^2}{\alpha\nu}b_x(L)^2\left[\frac{cothq_1L}{q_1} - \frac{cothq_2L}{q_2}\right]. \quad (254)$$

It is straightforward to show that the sum of the viscous and resistive dissipation is equal to the total energy input, given by (246), with V_1 given by (215).

It is evident from this example that there is no overall algebraic relation (i.e., no Ohms law) between $\mathbf{E}$ and $\mathbf{j}$ because the velocity $\mathbf{v}$ varies across the thickness of the ionosphere. There is (117), but it cannot be used until $\mathbf{v}(z)$ is known. So, as already noted, it is not possible to compute the current flow in the ionosphere from the electric field applied at the upper surface of the ionosphere without first solving the dynamical equations for $\mathbf{v}(z)$. The $\mathbf{B},\mathbf{v}$ paradigm begins with exactly that dynamical problem, and the solution of the dynamical equations provides both $\mathbf{E}(z)$ and $\mathbf{j}(z)$ if we wish to know them.

11. Equivalent Electric Circuit

The $\mathbf{E}$, $\mathbf{j}$ paradigm has led to the idea that various aspects of the magnetosphere can be represented by a fixed electric circuit, to which Kirchhoff's laws can be applied to deduce the dynamical properties of the magnetosphere. It is declared that $\mathbf{j}$ is channeled through the path of least ionospheric resistivity, and it is declared that a change in the parameters anywhere around the circuit necessarily effects the current flow everywhere else around the circuit, just as it would in a rigid electrical loop circuit of wires, resistors, inductances, applied emfs, etc. in the laboratory. Unfortunately, the magnetosphere is not a rigid body so that the circuit analogy has only limited use. The electric current is determined by the deformation $\nabla \times \mathbf{B}$ of the magnetic field, that is, by the force $\mathbf{F} = \mathbf{B} \times (\nabla \times \mathbf{B})$ exerted on the field by the gas with $(\nabla \times \mathbf{B})_\perp = \mathbf{F} \times \mathbf{B}/B^2$. This has no general tendency to coincide precisely with the region of minimum $(\eta + \beta)$. A more fundamental difficulty is created by the fact that the current path is distributed broadly over a range of motion $\mathbf{v}$, with varying local electric field $\mathbf{E}'(= \mathbf{E} + \mathbf{v} \times \mathbf{B}/c)$. There is no uniquely defined "voltage" to apply to the circuit. Consequently, different portions of the circuit behave more or less independently of each other, their connection being only the dynamical one described by (4) and (5) rather than by the simultaneity implied by Kirchhoff's laws. The analogous electric circuit can be constructed, if at all, only after the dynamical problem has been solved to determine $\mathbf{B}$ and $\mathbf{v}$.

The general apriori inapplicability of an electric circuit concept can be seen directly from the induction equation (5) and the momentum equation (4), plus whatever heat flow conditions are needed to specify p_{ij}. The two equations together determine the local dynamical evolution of $\mathbf{v}$ and $\mathbf{B}$. Given the initial $\mathbf{v}$ and $\mathbf{B}$ throughout the interior of a closed surface S, the subsequent dynamical evolution of $\mathbf{v}$ and $\mathbf{B}$ throughout the interior of S is uniquely determined by the values of $\mathbf{v}$ and $\mathbf{B}$ on S, regardless of what happens elsewhere along the current paths extending outward across S. And that is not a property of a local portion of a simple electric circuit, where the interruption of the current anywhere around the circuit instantaneously affects the current everywhere. One may introduce more complicated circuits including moving loaded lines to simulate Alfven wave propagation, of course, but such circuits become precise only in the limit of a continuous distribution of circuit branches. However the correct behavior of the continuous distribution of circuit branches is described exactly by (4) and (5) and cannot be determined by other means in the general case.

It is sufficient for present purposes of illustration of electric current flow to consider the simple case of a viscous nonconducting ($\eta = \infty$) atmosphere extending from the fixed surface $z = -\lambda$ to $z = 0$. In $0 < z < L$ there is an idealized ionosphere in which there is a resistive diffusion coefficient η and a uniform viscosity $\mu = \rho\nu$. Above $z = L$ there is only a tenuous highly conducting plasma ($\eta = 0$), extending up to $z = \Lambda$. The system is penetrated by a uniform magnetic field B in the z direction. The system is driven from above, as in section 10, by moving the foot points of the magnetic field B at the magnetopause ($z = \Lambda$) in the y direction, producing a y component $Bb(x, L)$ throughout the tenuous magnetosphere $L < z < \Lambda$, as in section 10. There is, however, an important difference and that is that in the present case it is assumed that $b(x, L)$ varies slowly with x, on a scale $l >> L, \Lambda$. The net effect is to drive the system in the y direction with the speed $v(x, z)$ under the applied Maxwell stress $B^2 b(x, L)/4\pi$. As a consequence of the slow variation of $b(x, L)$ with x, there is, in fact, a small field component in both the x direction and the z direction. However, they are small $O(L^2/l^2)$ compared to $b(x, L)$, so they can be neglected. Linearizing the dynamical equations for b and v, and neglecting all terms in v_x, b_x, and all terms of second order in $\partial/\partial x$ and v and b, the result for stationary flow is the velocity

$$v(x, z) = \frac{B^2 b(x, L)(\lambda + z)}{4\pi\rho\nu} \tag{255}$$

in the y direction throughout $-\lambda < z < 0$ so that $v(x, -\lambda) = 0$. In $0 < z < L$,

$$\frac{\partial v}{\partial z} + \eta \frac{\partial^2 b}{\partial z^2} = 0 \tag{256}$$

$$C^2 \frac{\partial b}{\partial z} + \nu \frac{\partial^2 v}{\partial z^2} = 0 \tag{257}$$

where C is again the Alfven speed $B/(4\pi\rho)^{\frac{1}{2}}$. A single integration yields

$$v + \eta \frac{\partial b}{\partial z} = V_2 \tag{258}$$

$$C^2 b + \nu \frac{\partial v}{\partial z} = V_4 \tag{259}$$

where V_2 and V_4 are constants in direct analogy with (190) and (192). The boundary conditions at $z = 0$ are $b = 0$, with the viscous stress equal to the applied Maxwell stress,

$$\rho\nu \frac{\partial v}{\partial z} = \frac{B^2 b(x, L)}{4\pi}, \tag{260}$$

and the velocity fitting to (255) at $z = 0$. At $z = L$, $b(x, z)$ becomes equal to the applied $b(x, L)$ while $\partial v/\partial z = 0$ since there is no viscous stress exerted on $z = L$ from $z > L$. It follows from (259) that the condition at $z = 0$ and $z = L$ both require

$$V_4 = C^2 b(x, L).$$

Then use (256) to eliminate $\partial v/\partial z$ from (259) obtaining

$$\frac{\partial^2 b}{\partial z^2} - k^2 b + k^2 b(x, L) = 0$$

where

$$k^2 = C^2/\nu\eta. \tag{261}$$

It follows that

$$b(x, z) = b(x, L)\left[1 - \frac{\sin hk(L - z)}{\sin hkL}\right] \tag{262}$$

$$v(x, z) = b(x, L) \left\{\frac{C^2\lambda}{\nu} + \eta k \left[\frac{\cos hkL - \cos hk(L - z)}{\sin hkL}\right]\right\} \tag{263}$$

with

$$V_2 = b(x, L)\left(\frac{C^2\lambda}{\nu} + \eta k \cot hkL\right).$$

in order to satisfy (255) at $z = 0$. The electric current is

$$j_x = -\frac{c}{4\pi}B\frac{\partial b}{\partial z},$$

$$= -\frac{c}{4\pi}\frac{b(x,L)k}{sin\ hkL}cos\ hk(L-z), \quad (264)$$

$$j_y \cong 0, \quad (265)$$

$$j_z = \frac{cB}{4\pi}\frac{\partial b}{\partial x},$$

$$= \frac{cB}{4\pi}\frac{\partial b(x,L)}{\partial x}\left[1 - \frac{sin\ hk(L-z)}{sin\ hkL}\right]. \quad (266)$$

Thus j_z is small $O(1/kl)$ compared to j_x. Note, then, that the principal current j_x declines upward across $0 < z < L$ in response to the declining Lorentz force while the resistivity η is uniform. This illustrates the point made earlier that the ionospheric currents are not simply concentrated at the level where η is a minimum. There are other influences as well,for example the variation of $\mathbf{E}'$ with the fluid velocity $\mathbf{v}$. Appendix E provides an illustrative example of the effect of the vertical variation of $\eta(z)$, showing again that there is no direct local relation between $j_x(z)$ and $\eta(z)$ alone.

Now the principal electric current is in the negative x direction, providing a one-dimensional electric circuit extending (if for the moment we neglect the slow variation with x) from $x = -\infty$ to $x = +\infty$. If may be imagined that the space is periodic, so that the circuit closes at $x = \pm\infty$. The total current J_x per unit length in the y direction is

$$J_x = \int_0^L dz j_x$$

$$= -\frac{cBb(x,L)}{4\pi} \quad (267)$$

which is simply an expression of Ampere's law. The electric field E_x is readily computed from Ohm's law, which is written $j_x = \sigma E'_x(\sigma = c^2/4\pi\eta)$ and E'_x is the electric field

$$E'_x = E_x + v(x,z)B/c \quad (268)$$

in the frame of reference of the fluid. Then E'_x follows as j_x/σ, given by (264). The terms depending upon z cancel out of this relation, of course, because $\nabla \times \mathbf{E} = 0$, leaving the constant terms,

$$E_x = -\frac{Bb(x,L)}{c}\left\{\frac{C^2\lambda}{\nu} + \eta k cot\ hkL\right\}. \quad (269)$$

If the fixed circuit analogu were correct in some direct way, one would expect to find $j_x = \sigma E_x$. However, of course, this is not correct, because the fluid is moving and the relevant electric field for driving the current is E'_x, as noted in sections 9 and 10. So even in so simple a system as the one-dimensional circuit constructed here, the construction can be carried through only after the problem is solved.

However, now consider the consequences of the slow variation of the driving field $b(x, L)$ with x. The result is a small j_x, hitherto neglected, flowing upward from $z = L$ along the perturbed field and closing at $z = \Lambda$ where the driving force that causes the Maxwell stress $B^2b(z, L)/4\pi$ is applied. Suppose, as an example, that

$$b(x,L) = b_o + b_1 cos\ Kx \quad (270)$$

where $b_1 < b_o$ and $K(= 1/l)$ is small compared to k. Then

$$j_x = -\frac{cBb_1K}{4\pi}sin\ Kx\left[1 - \frac{sin\ hk(L-z)}{sin\ hkL}\right] \quad (271)$$

The result is a series of side loops sketched in Figure 8. The conventional electric circuit analog can be aware of these auxiliary loops only by considering the stress balance, and it can treat them quantitatively only after carrying through the above formal solution to the dynamical problem, or its equivalent.

As a final point, note that for $b(x, L) << 1$, treated here, the magnetic conditions at any position x depend only on the local $b(x, L)$. The small field perturbation elsewhere has no first-order effect on conditions at x. Thus the idea that a change in conditions at one location in an electric circuit impacts conditions everywhere around the circuit is inapplicable, showing that the circuit analog conveys the wrong physics. Local conditions are exactly that, and

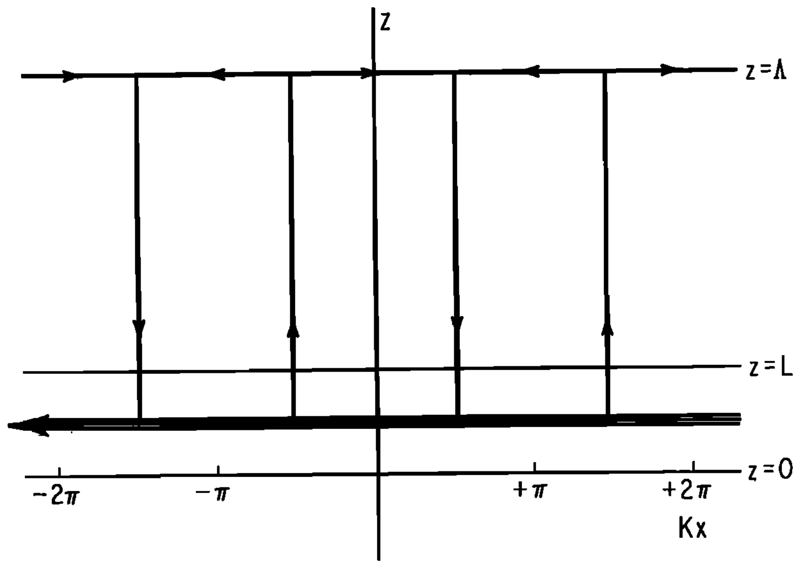

Figure 8. A schematic drawing of the electric circuit analogue with the main current described by (264) and the auxiliary current loops indicated by equation (271).

they sprout their own auxiliary current loops as needed to remain localized.

In the more complex situation when the applied force is strongly inhomogeneous and a function of time, the various parts of the system communicate with each other only on the Alfven wave transit time, which may be slow compared to the characteristic rate at which the applied force changes. The instantaneous circuit is entirely inappropriate and some suitable loaded line would be necessary to represent the effect. However, again a correct circuit analog can be set up only after the appropriate field equations for $\mathbf{v}$ and $\mathbf{B}$ are properly solved. That is to say, the activity of the magnetosphere is mechanical in nature, and electric circuit analogs can be constructed only after the fact. The question, then, is what is to be gained by their construction?

In conclusion, the solution to a problem in magnetospheric plasma lies in the dynamical stress and momentum balance and the associated deformation of $\mathbf{B}$. Electric circuit analogies can be constructed once the problem is solved, but there seems to be little to be gained by it since the electric circuit deals only with the peripheral quantities $\mathbf{E}$ and $\mathbf{j}$, which are directly available from $\mathbf{B}$ and $\mathbf{v}$.

12. Concluding Remarks

There is little to say in conclusion, unless we are to repeat the general remarks in the introductory section. The idealized examples making up the bulk of the text illustrate the deductive method of the $\mathbf{B}$, $\mathbf{v}$ paradigm and the general applicability to the macro physics of the magnetosphere. The paradigm follows directly from the equations of Newton and Maxwell, yielding a set of partial differential equations that are self-consistent and complete in themselves. Thus the $\mathbf{B}$, $\mathbf{v}$ paradigm works directly with the mechanicial pushing and pulling that makes up macroscopic magnetospheric activity, thereby providing a direct deductive approach to the dynamics of the magnetosphere, and avoiding the difficulties of the $\mathbf{E}$, $\mathbf{j}$ paradigm, which focuses on secondary quantities.

As the subject of magnetospheric physics advances into more complex dynamical problems, the deductive approach becomes imperative. The principles and declarations that carry the $\mathbf{E}$, $\mathbf{j}$ paradigm forward are already in some degree of error in the simple examples cited in the text: The definition of parallel currents in terms of the unperturbed field, the idea of the dynamical inward penetration of the interplanetary electricfield along the magnetic field lines, and the notion that simple electric circuit analogs define magnetospheric dynamics before the fact, are all cases in point. It can be argued, of course, that the $\mathbf{E}$, $\mathbf{j}$ paradigm can be made to work with sufficient care and foresight. However, this is equivalent to the statement that, if the correct solution to a dynamical problem is known in some way, then the description of the dynamics is easily translated into the $\mathbf{E}$, $\mathbf{j}$ paradigm, which we have repeatedly emphasized. The essential point is that the $\mathbf{B}$, $\mathbf{v}$ paradigm, with its tractable and complete set of partial differential equations, provides the means for deducing the correct dynamical solution from the boundary conditions and the basic physical principles embodied in Newton's and Maxwell's equations. Only with a purely deductive procedure can one proceed with assurance of a correct result for a specified circumstance.

Appendix A: Path of Flux Bundles

In polar coordinates (ϖ, φ) Fermat's principle can be written

$$\delta \int d\varpi \frac{(1+\varpi^2\varphi'^2)^{\frac{1}{2}}}{\varpi^2} = 0$$

where $\varphi' \equiv d\varphi/d\varpi$ and the index of refraction B is $1/\varpi^2$. Since φ does not appear in the integrand, Euler's equation reduces to

$$\frac{d}{d\varpi}\left[\frac{\varphi'}{(1+\varpi^2\varphi'^2)^{\frac{1}{2}}}\right] = 0.$$

Integration yields

$$\varphi'^2(1-k^2\varpi^2) = k^2$$

where k is an arbitrary constant. If $k = 0$, the result is the family of radial lines, for which $\varphi' = 0$. In the opposite extreme that φ' is large, it follows that $\varpi = \lambda = const$, providing circles concentric about the origin. For $0 < k\varpi < 1$, integration yields

$$k\varpi = sin(\varphi - \varphi_1)$$

where φ_1 is an arbitrary constant. This provides a circle of diameter $1/k$ through the origin. The center of the circle lies at $\varpi = 1/2k$, $\varphi = \varphi_1 + \frac{1}{2}\pi$.

Appendix B: Angular Deflection of Flux Bundle

The field lines of the ambient two-dimensional dipole are given by the circles

$$(y-\tfrac{1}{2}a)^2 + z^2 = \tfrac{1}{4}a^2, \tag{B1}$$

for which

$$\frac{dz}{dy} = -\frac{y-\frac{1}{2}a}{z}. \tag{B2}$$

The diameter of the circle is a and the center is located on the y axis at $y = \frac{1}{2}a$. The perturbed field line, described by (71), can be written

$$(y-\tfrac{1}{2}a')^2 + (z-\Delta b)^2 = \tfrac{1}{4}a'^2 \tag{B3}$$

where $a' \equiv a + \Delta a$ represents the diameter, with the center at $y = \frac{1}{2}a'$, $z = \Delta b$, with Δa and Δb given by (72) and $-\frac{1}{2}a\vartheta$, respectively. It follows that

$$\frac{dz}{dy} = -\frac{(y-\frac{1}{2}a')}{z-\Delta b}. \tag{B4}$$

We are interested in the angular difference in the direction of the two circles where they intersect at some point (y, z). Let y specify the location of the point of intersection, for which $z(> 0)$ follows as

$$z = +(ay-y^2)^{\frac{1}{2}}$$

from (B1). Substituting this result into (B3) yields

$$a \cong a' + 2\Delta b\left(\frac{a-y}{y}\right)^{\frac{1}{2}} \tag{B5}$$

in terms of the location y of the point of intersection.

The slope $(dz/dy)_a$ of the ambient field line and $(dz/dy)_p$ of the perturbed field line follow from (B2) and (B4). Their difference is

$$\left(\frac{dz}{dy}\right)_a - \left(\frac{dz}{dy}\right)_p = -$$

$$\frac{\frac{1}{2}(a-a')[y(a-y)]^{\frac{1}{2}} - \Delta b(y-\frac{1}{2}a)}{[y(a-y)]^{\frac{1}{2}}\{[y(a-y)]^{\frac{1}{2}} - \Delta b\}} \quad \text{(B6)}$$

$$\cong -\Delta b \frac{\frac{1}{2}a}{y(a-y)} \quad \text{(B7)}$$

upon neglecting Δb compared to $y^{\frac{1}{2}}(a-y)^{\frac{1}{2}}$.

Denote the small angle between the two slopes by ψ. Let

$$\left(\frac{dz}{dy}\right)_a = tan\theta, \left(\frac{dz}{dy}\right)_p = tan(\theta+\psi).$$

Since

$$tan(\theta+\psi) - tan\theta \cong \psi/cos^2\theta, \quad \text{(B8)}$$

it follows from (B7) and (B8) that

$$\psi \cong -2\Delta b/a$$

$$\cong \vartheta. \quad \text{(B9)}$$

So the deflection ϑ relative to the ambient field is independent of position along the perturbed field line and has the same value as given by (63).

Appendix C: Ionosphere Without Hall Effect $\alpha = 0$:

As another example of the diffusion of field in a motionless ionosphere with $\alpha = 0$, suppose that the field or tilt applied at $z = L$ is

$$b_x(L,t) = b_o\left[erfc\left(\frac{L}{[(\eta+\beta)t]^{\frac{1}{2}}}\right) + \left[\frac{(\eta+\beta)t}{\pi L^2}\right]^{\frac{1}{2}}\left\{1 - exp\left[-\frac{L^2}{(\eta+\beta)t}\right]\right\}\right].$$

Then for $(\eta+\beta)t << L^2$,

$$b_x(L,t) \sim b_o\left[\frac{(\eta+\beta)t}{\pi L^2}\right]^{\frac{1}{2}}$$

with the applied field increasing as $t^{\frac{1}{2}}$ from zero at time $t = 0$. At large t,

$$b_x(L,t) \sim b_o\left\{1 - \left[\frac{L^2}{\pi(\eta+\beta)t}\right] + \cdots\right\}$$

The field applied to the upper surface $(z = L)$ of the ionosphere grows initially as $t^{\frac{1}{2}}$ and subsequently, after a time comparable to the characteristic diffusion time $L^2/(\eta+\beta)$, levels off to a uniform value b_o. The field throughout the ionosphere $0 < z < L$ follows from (153) as

$$b_x(z,t) = \frac{b_o}{2L}\left[(L+z)erfc\left\{\frac{L+z}{[4(\eta+\beta)t]^{\frac{1}{2}}}\right\} - (L-z)erfc\left[\frac{L-z}{[4(\eta+\beta)t]^{\frac{1}{2}}}\right\}\right] + \left[\frac{4(\eta+\beta)t}{\pi}\right]^{\frac{1}{2}}\left\{exp\left[-\frac{(L-z)^2}{4(\eta+\beta)t}\right] - exp\left[-\frac{(L+z)^2}{4(\eta+\beta)t}\right]\right\}\Big].$$

Then for $4(\eta+\beta)t >> L^2$,

$$b_x(z,t) \sim b_o\frac{z}{L}\left[1 - \frac{L}{[\pi(\eta+\beta)t]^{\frac{1}{2}}}\cdots\right].$$

The field relaxes to a linear increase upward through the ionosphere.

As a final example, suppose that the field applied at $z = L$ grows linearly with time in the beginning but after a time comparable to the characteristic diffusion time $L^2/(\eta+\beta)$, slacks off to an increase proportional only to $t^{\frac{1}{2}}$. Let

$$b_x(L,t) = b_o\frac{\eta t}{L^2}\left\{1 + \frac{2L}{[\pi(\eta+\beta)t]^{\frac{1}{2}}}exp\left[-\frac{L^2}{(\eta+\beta)t}\right] - \left[1 + \frac{2L^2}{(\eta+\beta)t}\right]erfc\frac{L}{[(\eta+\beta)t]}^{\frac{1}{2}}\right\}.$$

For $(\eta+\beta)t << L^2$,

$$b_x(L,t) \sim b_o\frac{\eta t}{L^2},$$

while for $(\eta+\beta)t >> L^2$

$$b_x(L,t) \sim b_o\left[\frac{(\eta+\beta)t}{\pi L^2}\right]^{\frac{1}{2}}.$$

The field throughout $0 < z < L$ follows as

$$b_x(z,t) = \frac{b_o(\eta+\beta)t}{L^2}\left\{\left[\frac{(L-z)^2}{2(\eta+\beta)t} + 1\right]erfc\frac{L-z}{[4(\eta+\beta)t]^{\frac{1}{2}}} - \left[\frac{(L+z)^2}{2(\eta+\beta)t} + 1\right]erfc\frac{L+z}{[4(\eta+\beta)t]^{\frac{1}{2}}} + \frac{L+z}{[\pi(\eta+\beta)t]^{\frac{1}{2}}} exp\left[-\frac{(L+z)^2}{4(\eta+\beta)t}\right] - \frac{L-z}{[\pi(\eta+\beta)t]^{\frac{1}{2}}}exp\left[-\frac{(L-z)^2}{4(\eta+\beta)t}\right]\right\}.$$

Then for $(\eta+\beta)t >> L$,

$$b_x(z,t) \sim b_o\frac{4}{L^2}\left[\frac{(\eta+\beta)t}{\pi}\right]^{\frac{1}{2}}z\left\{1 - \left[\frac{\pi}{(\eta+\beta)t}\right]^{\frac{1}{2}}\frac{L}{2}\right\}$$

and again $b_x(z,t)$ increases linearly upward across the ionosphere. The linear variation is a consequence of the vanishing or declining time rate of increase of $b_x(L,t)$ in the limit of large time t. The local Lorentz force in the ionosphere is proportional to $\partial b/\partial z$ and is asymptotically uniform over z.

Appendix D: Polynomial Solutions to Equations (9.51) and (9.54)

It is a simple matter to generate polynomial solutions to (183) and (186). Generally, the solutions can be written as infinite series, but with proper choice of a, the series can be terminated at any desired point. Starting with the simplest solutions, consider $\Psi = \Psi_o$, satisfying (183) for $a = 0, 1$. The resulting solutions are

$$b_x = D_o, \; b_y = D_1 + D_2 z, \tag{D1}$$

$$b_x = D_o t, \; b_y = D_o z^2/2\alpha, \tag{D2}$$

where D_o, D_1, and D_2 are arbitrary constants. The complementary solutions are obtained by interchanging b_x and b_y and changing the sign of one of them.

If $\Psi = \xi$, then $a = \frac{1}{2}, \frac{3}{2}$. The solutions are

$$b_x = z, \; b_y = 0, \tag{D3}$$

$$b_x = zt, \; b_y = z^3/6\alpha. \tag{D4}$$

If $\Psi = \xi^2$, then $a = 1, 2$. The solutions are

$$b_x = z^2, \; b_y = -2\alpha t, \tag{D5}$$

$$b_x = z^2 t, \; b_y = z^4/12\alpha - \alpha t^2. \tag{D6}$$

The solution (D5) is just the complementary solution to (D2). If $\Psi = \xi^3$, then $a = \frac{3}{2}, \frac{5}{2}$. The solutions are

$$b_x = z^3, \; b_y = -6\alpha z t, \tag{D7}$$

$$b_x = z^3 t, \; b_y = z^5/20\alpha - 3\alpha z t^2. \tag{D8}$$

The solution (D7) is the solution complementary to (D4).

For higher order polynomials, write

$$\Psi = \xi^n + a_1 \xi^{n+4} + a_2 \xi^{n+8} + \cdots. \tag{D9}$$

Substitution into (182) yields the indicial equation

$$n(n-1)(n-2)(n-3) = 0, \tag{D10}$$

so that $n = 0, 1, 2, 3$. Then

$$a_1 = -\frac{a^2 - a(n+1) + \frac{1}{4}n(n+2)}{\alpha^2(n+4)(n+3)(n+2)(n+1)}, \tag{D11}$$

$$a_2 = -a_1 \frac{a^2 - a(n+5) + \frac{1}{4}n(n+4)(n+6)}{\alpha^2(n+8)(n+7)(n+6)(n+5)}, \tag{D12}$$

$$a_{m+1} = -a_m \times \frac{a^2 - a(n+4m+1) + \frac{1}{4}(n+4m)(n+4m+2)}{(n+4m)(n+4m-1)(n+4m-2)(n+4m-3)}. \tag{D13}$$

The series terminates after the $m+n$ term with the choice

$$a = \tfrac{1}{2}n + 2m \pm \tfrac{1}{2}. \tag{D14}$$

The linearity of (183) permits superposition of these many individual forms, of course.

In a similar vein polynomial solutions of (185) can be generated. For instance, the solution $\Psi \sim \xi$ satisfies (185) for $a = \frac{1}{2}, \frac{3}{2}$, giving rise to the solutions z and (186) and (187). The quadratic form $\Psi \sim \xi^2 + Q$, where Q is a constant yields $\Psi = \xi^2 + 4(\eta+\beta)$ for $a = 1$, and $\Psi = \xi^2 + 2(\eta+\beta)$ for $a = 2$. Hence for $a = 1$,

$$b_x = z^2 + 4(\eta+\beta)t. \tag{D15}$$

Substituting this into (172) and (173) yields

$$b_y = \frac{\eta+\beta}{\alpha} z^2 - 2\alpha\left[1 - \left(\frac{\eta+\beta}{\alpha}\right)^2\right]t \tag{D16}$$

together with the complementary solutions. For $a = 2$,

$$b_x = z^2 t + 2(\eta+\beta)t^2 \tag{D17}$$

with

$$b_y = \frac{z^4}{12\alpha} + \frac{(\eta+\beta)z^2 t}{\alpha} - \alpha\left[1 - \left(\frac{\eta+\beta}{\alpha}\right)t^2\right] \tag{D18}$$

etc. Note, however, that these solutions do not individually or in combination satisfy the boundary condition that $b_x = b_y = 0$ at $z = 0$.

The higher-order polynomial solutions can be generated along the lines indicated for (182).

Appendix E: Cross Field Currents

The cross-field currents are determined by the force exerted on the field by the fluid, so that the cross field currents can be determined only after consideration of the dynamical equations. Therefore there is no simple algebraic relation between the cross field current density and the local resistive diffusion coefficient, although in the final asymptotic steady state of ionospheric motion driven by Maxwell stresses transmitted downward by a deformed magnetic field, the Lorentz force depends in a functional way upon the form of the vertical variation of the resistive diffusion coefficient $\eta(z)$. To treat a single simple illustrative example, consider the case taken up in section 11 ignoring the slow variation with x, which is of no interest in the present case. Write $b(z)$ in place of $b(x,z)$ and imagine that again the motion in the y direction at the underside $z = 0$ of the magnetosphere has some such value $v(0)$ as given by (255) with $\nu = \nu(0)$ throughout $-\lambda < z < 0$. With ρ, ν and η as functions of height z throught out the ionosphere $0 < z < L$, rewrite (256) and (257) as

$$\frac{dv}{dz} + \frac{d}{dz}\eta\frac{db}{dx} = 0, \; \frac{B^2}{4\pi}\frac{db}{dz} + \frac{d}{dz}\rho\nu\frac{dv}{dz} = 0. \tag{E1}$$

Integration over z yields

$$v + \eta\frac{db}{dz} = V_2, \; \frac{B^2}{4\pi}b + \rho\nu\frac{dv}{dz} = V_4 \tag{E2}$$

where V_2 and V_4 are independent of z, with V_4 again equal to the

Maxwell stress $B^2 b(L)/4\pi$, as required by the boundary conditions at $z = L$ where the viscous stress falls to zero. Introduce the variable

$$\zeta(z) = \int_0^z dz'/\eta(z') \quad \text{(E3)}$$

with the dimensions of inverse velocity and define the function $f(\zeta)$ as $[\rho(z)\nu(z)/\rho(0)\nu(z)]^{\frac{1}{2}}$, with $C^2 \equiv B^2/4\pi\rho(0)$ again. Equation (E2) becomes

$$v + db/d\zeta = V_2, C^2 b + f^2 dv/d\zeta = C^2 b(L) \quad \text{(E4)}$$

from which it follows that

$$\left[\frac{d^2}{d\zeta^2} - \frac{C^2}{f^2(\zeta)}\right][b(z) - b(L)] = O. \quad \text{(E5)}$$

In the simplest case suppose that $f(\zeta)$ is a constant. Then

$$b(z) = b(L)\left\{1 - \frac{\sinh\frac{C}{f}[\zeta(L) - \zeta(z)]}{\sinh\frac{C}{f}\zeta(L)}\right\}. \quad \text{(E6)}$$

The velocity follows from (E4) and the boundary condition at $z = 0$ from (255), with the result

$$v(z) = Cb(L)\left\{\frac{C\lambda}{\nu(0)} + \frac{1}{f}\right.$$

$$\left.\left[\coth\frac{C\zeta(L)}{f}\left(1 - \cosh\frac{C\zeta(z)}{f}\right) + \sinh\frac{C\zeta(z)}{f}\right]\right\}. \quad \text{(E7)}$$

The cross field current is

$$j_x(z) = -\frac{c}{4\pi}B\frac{db}{dz}$$

$$= -\frac{cBb(L)C}{4\pi f\eta(z)}\frac{\cosh\frac{C}{f}[\zeta(L) - \zeta(z)]}{\sinh\frac{C}{f}\zeta(L)}. \quad \text{(E8)}$$

The electric field follows from Ohm's law as

$$E_n = \frac{4\pi}{c^2}\eta(z)j_x(z) - \frac{B}{c}v(z)$$

$$= \frac{Bb(L)C}{c}\left[\frac{1}{f}\coth\frac{C\zeta(L)}{f} - \frac{C\lambda}{\nu(o)}\right]. \quad \text{(E9)}$$

A weak ionosphere, $C\zeta(L)/f << 1$, yields

$$b(z) \cong b(L)\zeta(z)/\zeta(L) \quad \text{(E10)}$$

$$v(z) = Cb(L)\left\{\frac{C\lambda}{\nu(o)} + \frac{C\zeta(z)}{f}\left[1 - \frac{\zeta(z)}{\zeta(L)}\right] + \cdots\right\} \quad \text{(E11)}$$

$$j_x = -\frac{cBb(L)}{4\pi\zeta(L)\eta(z)}\left\{1 + \frac{c^2}{f^2}[\zeta(L) - \zeta(z)]^2 + \cdots\right\} \quad \text{(E12)}$$

$$E_x = \frac{Bb(L)C}{c}\left[\frac{1}{C\zeta(L)} - \frac{C\lambda}{\nu(o)}\right\}. \quad \text{(E13)}$$

The velocity varies but little across the ionosphere and j_n is directly proportional to $1/\eta(z)$ to lowest order as a consequence.

On the other hand, the terrestrial ionosphere is not weak in this respect. Including the Pedersen resistive diffusion

$$\beta \cong \frac{B^2\tau_i}{4\pi nM} \quad \text{(E14)}$$

(given by (115) with $m/\tau_e << M/\tau$) along with η, write

$$\tau_i = \lambda_i/w_i$$

$$= 1/NA_i w_i \quad \text{(E15)}$$

where w_i is the ion thermal velocity, and A_i is the cross section for collision of an ion with a neutral atom. The Alfven speed C is essentially $B/(4\pi NM)^{\frac{1}{2}}$ where M is taken to be the mass of a neutral air atom as well as an ion. Approximate the kinematic viscosity ν by $\frac{1}{3}\lambda w_i$ where λ is the mean free path $1/NA$ for collision of neutral atoms with neutral atoms, with a cross section A. Then with $\rho(z)/\rho(0) \sim 1$, it follows that $f^2 = \nu/\beta(\beta >> \eta)$. Putting $\zeta(L) = L/\beta$, it follows that, in order of magnitude,

$$\frac{C\zeta(L)}{f} \cong \frac{CL}{(\nu\beta)^{\frac{1}{2}}}$$

$$= (3L^2 nNA_i A)^{\frac{1}{2}}. \quad \text{(E16)}$$

It is evident at once that the effect declines upward as $(nN)^{\frac{1}{2}}$, so that the ionospheric D layer appears to be the principal contributor. Then with $n = 10^5$ electrons /cm^3 and $N = 10^{12}$ atoms /cm^3, $A_i = A = 10^{-16}$cm^2, and the characteristic scale $L = 10^7$ cm for the D region the result is approximately 0.5. However, the cross section for ions on neutral atoms is substantially larger than the geometrical atomic dimension because of the deformation of the neutral atom by the electrostatic field of the passing ion. At the low velocities ($w_i \lesssim 10^5$cm/s) in the D region ($T \sim 500K$) A_i may be 10^{-14} cm ^{2} or larger. The value of $C\zeta(L)/f$ is ten or more times larger, that is 5 or more. The result is a strong modulation of $\dot{\eta}(z)^{-1}$ by the factor $\cosh[C[\zeta(L) - \zeta(z)]/f]/\sinh[C\zeta(L)/f]$ on the right hand side of (E8).

It is easy to construct more realistic examples. Instead of $f = const$, the choice $f \sim \zeta^a$ or exp $\kappa\zeta$ yields solutions to (E5) in terms of Bessel functions with imaginery arguments, etc. However, the essential point is the same, that $C\zeta(L)/f \gtrsim 0(1)$ so that the velocity varies strongly upward through the D layer, with the result that j_x does not vary simply as $[\eta(z) = \beta(z)]^{-1}$ upward across the ionosphere.

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