Transformation Between ECL50 and System III Co-ordinate Systems

Logan Dougherty

General Rotation Matrix

- In General, for a 1x3 Column vector, there exists a 3x3 Rotation matrix that will rotate the vector to a new co-ordinate system
- The Rotation matrix can be a combination of three rotations about three axes, which in turn produces a rotation matrix that has 9 unknown coefficients.

Position Transformation

- In general, the transformation can be written as follows.
- $$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
- Where x, y, and z are the original co-ordinates and x', y', and z' are the transformed ones.
- The c values are the Rotation matrix.

Velocity Transformation

- In the Voyager I and II data, there also happens to be 2 vector quantities.
- As well as the position, there is also velocity, so it holds that,
- $$\begin{bmatrix} {v_x}' \\ {v_y}' \\ {v_z}' \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$
- This now provides 6 equations for the 9 unknown values in the rotation matrix.

Necessity for a Third Equation

- There needs to be another set of three equations to uniquely determine the Rotation matrix.
- Let's define a quantity similar to angular momentum:
- $\vec{p} = \vec{r} \times \vec{v}$
- Because both co-ordinate systems are orthogonal and share an origin, then this momentum like term should also be transformed with the same exact Rotation matrix, giving a total of 9 equations and 9 unknowns.

Combining the Equations

- The 9 equations can then be rewritten into one compact matrix form:
- $$\begin{bmatrix} x' & v_x' & p_x' \\ y' & v_y' & p_y' \\ z' & v_z' & p_z' \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} x & v_x & p_x \\ y & v_y & p_y \\ z & v_z & p_z \end{bmatrix}$$
- Now there are 9 equations and 9 unknowns so it is solvable.
- Choosing to rewrite it:
- B = R A , where R is the rotation matrix and B and A are the matrices above.

Solving for R

- If B = R A, then it is simple to solve for R using a computer and taking the inverse of A.
- $R\ A\ A^{-1} = B\ A^{-1}$
- Because $A\ A^{-1} = 1$, this produces the matrix for R.
- $\begin{bmatrix} x' & {v_x}' & {p_x}' \\ y' & {v_y}' & {p_y}' \\ z' & {v_z}' & {p_z}' \end{bmatrix} \begin{bmatrix} x & v_x & p_x \\ y & v_y & p_y \\ z & v_z & p_z \end{bmatrix}^{-1} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$

Transforming the other way

- It is simple to get the transformation matrix for going from B to A as well.
- If $A = R_2 B$ then $R_2 = A\,B^{-1}$
- Denoting the elements of R_2 with d instead of c:
- $\begin{bmatrix} x & v_x & p_x \\ y & v_y & p_y \\ z & v_z & p_z \end{bmatrix} \begin{bmatrix} x' & v_x{}' & p_x{}' \\ y' & v_y{}' & p_y{}' \\ z' & v_z{}' & p_z{}' \end{bmatrix}^{-1} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}$

- Thus, given two vectors in each co-ordinate system, and the fact that the co-ordinate systems are both orthogonal and share a common origin, it is possible to uniquely determine a 3x3 Rotation matrix between the two co-ordinate systems.