VOYAGER MEMORANDUM 135 (Revised)

From Fran Bagenal January 9, 198-

To: Voyager PLS internal/JPL MAGPAC team

From: Fran Bagenal

Subject: **Calculating Plasma Densities in the Io Plasma Torus**

Date: January 7, 1987

I have developed the following method for calculating the density of ion and electron species at a given point in the Io plasma torus.

INPUT: Ion and electron densities, temperatures and thermal anisotropies ($T_\perp/T_\parallel$) at the centrifugal equator as a function of radial distance.

ASSUME: (1) The temperature of each species is assumed to remain constant along a field line. The plasma is then assumed to be distributed with latitude along a field line in accordance with ambipolar diffusive equilibrium, i.e., we find the ambipolar electric field that balances the centrifugal and pressure gradient forces on the plasma. For a plasma consisting of only one ion (mass M_i, charge Z_i) and one electron species (with T_e=T_i) the resulting density distribution can be found (analytically) to be a simple exponential scale height for small distances z from the centrifugal equator.

$$n = n_o \exp-(z/H)^2$$

where $H=[2kT_i(Z_i+1)/3m_i\Omega^2]^{1/2}$. For a multi-ion plasma we have to determine the ambipolar electric field numerically.

(2) Until we have further information on the azimuthal distribution of plasma in the Io torus the variation with longitude must be assumed to be constant (azimuthal symmetry) or modeled with, for example, a simple sinusoidal variation peaking, at say, λ_{III} = 200 degrees (c.f. models of Trauger or Morgan).

OUTPUT: Density for each ion species as functions of L-shell, centrifugal latitude and λ_{III}. Thus with the measured radial variation, the assumed longitudinal variation and the calculated latitudinal variation we can build a three-dimensional model.

LIMITATIONS: Since the assumption that temperature remains constant along a field line is only really valid for low latitudes this model should not be relied on for densities above ~20° latitude. Of course, above ~33° the field lines bend over (i.e., z decreases with θ). Closer to the planet gravity will become important. But the most serious problem is the issue of calculating bulk properties (i.e., density and temperature) in a region populated mainly by non-thermal particles. Nevertheless, the volume of the flux tube decreases as $\cos^7\theta$ so that the high latitude plasma contributes proportionally less to integral quantities (e.g., NL^2 or average T_i, etc.) which are the important quantities as far as the physics is concerned.

BACKGROUND: (See Vasyliunas' Chapter 11 of Dessler's "Physics of the Jovian Magnetosphere" or references in the appendix of Bagenal and Sullivan, J. Geophys. Res., *86*, 8447, 1981). The idea is to find the distribution of plasma along a field line that satisfies the *i* equations of field-aligned force balance for the *i* species:

$$(\nabla \cdot \vec{P})_{\parallel} + n_i\, m_i \left[\frac{d\mathbf{V}}{dt}\right]_{\parallel} + n_i\, Z_i\, q\, \mathbf{E}_{\parallel} = 0$$

[1] [2] [3]

where [1] consists of two terms

$$\nabla_{\parallel} P_{\parallel} - (P_{\parallel} - P_{\perp})\left[\frac{(\mathbf{B}\cdot\nabla)\mathbf{B}}{B^2}\right] = \frac{\partial P_{\parallel}}{\partial S} - (P_{\parallel} - P_{\perp})\frac{1}{B}\frac{\partial B}{\partial S}$$

where the second term is often called the magnetic mirror force which is zero for an isotropic plasma ($P_{\parallel} = P_{\perp}$). In steady state conditions [2] is the centrifugal force

$$n_i\, m_i \left[\frac{d\mathbf{V}}{dt}\right]_{\parallel} = -n_i\, m_i \frac{\partial}{\partial S}\left[\frac{1}{2}\Omega^2 \rho^2\right]$$

The third term [3] is the ambipolar electric force

$$n_i\, Z_i\, q\, \mathbf{E}_{\parallel} = n_i\, Z_i\, q \frac{\partial \phi}{\partial S}$$

where ϕ is the electric potential. Thus the top equation becomes:

$$\frac{\partial P_{\parallel}}{\partial S} - (P_{\parallel} - P_{\perp})\frac{1}{B}\frac{\partial B}{\partial S} - n_i\, m_i \frac{\partial}{\partial S}\left[\frac{1}{2}\Omega^2 \rho^2\right] + n_i\, Z_i\, q \frac{\partial \phi}{\partial S} = 0$$

If we assume $P_{\parallel} = n_i\, T_{\parallel}$ (where T is in eV) and $T_{\parallel}$ is constant along a field line, then

$$\frac{1}{n_i}\frac{\partial n_i}{\partial S} = \frac{(T_{\parallel} - T_{\perp})}{T_{\parallel}}\frac{1}{B}\frac{\partial B}{\partial S} + \frac{m_i}{T_{\parallel}}\frac{\partial}{\partial S}\left[\frac{1}{2}\Omega^2 \rho^2\right] - \frac{Z_i\, q}{T_{\parallel}}\frac{\partial \phi}{\partial S}$$

or

$$\frac{\partial}{\partial S}\log n_i = \frac{\partial}{\partial S}\left[\left[1 - \frac{T_{\perp}}{T_{\parallel}}\right]\log B + \frac{1/2\, m_i\, \Omega^2 \rho^2}{T_{\parallel}} - \frac{Z_i\, q\, \phi}{T_{\parallel}}\right]$$

Integrating along the field line from a reference point S_o to S we have

$$n_i(S) = n_i(S_o)\exp\left[\left[1 - \frac{T_{\perp}}{T_{\parallel}}\right]\log\frac{B(S)}{B(S_o)} + \frac{1/2\, m_i\, \Omega^2\, \{\rho^2(S) - \rho^2(S_o)\}}{T_{\parallel}} - \frac{Z_i\, q\, \{\phi(S) - \phi(S_o)\}}{T_{\parallel}}\right]$$

[1] [2] [3]

Translating this into quantities that we calculate (and swopping terms [1] ans [3])

$$n_i(S) = n_i(S_o)\exp\left[Z_i\, \mathrm{P}\,\frac{\mathrm{ANIS}_i}{T_i} + \mathrm{FACTOR}\, A_i\, \mathrm{FCENT}\frac{\mathrm{ANIS}_i}{T_i} - [\mathrm{ANIS}_i - 1]\,\mathrm{FMAG}\right]$$

where

$\mathrm{ANIS}_i = T_{\perp}/T_{\parallel}$

$T_i = T_{\perp} = T_{measured}$ (in eV)

P = ϕ = ambipolar potential, normalised to S_o (where P $\equiv$ 0).

Note P will have negative values at latitudes less than S_o.

FACTOR $= 1/2\, m_p\, \Omega^2 R_J^2 / q$

$= 0.825$

= conversion of the centrifugal potential FCENT to eV.

FCENT $= \rho^2(S) - \rho^2(S_o)$

$= R^2 \cos^2\theta_G - R_o^2 \cos^2\theta_{Go}$

= the centrifugal potential

varies with the cylindrical distance from the rotation axis, normalised to S_o.

FMAG $= \log \{B(S)/B(S_o)\}$

= the magnetic field magnitude relative to the field at S_o.

For a dipole magnetic field

$$\log \{B(S)/B(S_o)\} = \log \left[\left[\frac{1 + 3\sin^2\theta_o}{1 + 3\sin^2\theta} \right]^{1/2} \left[\frac{\cos\theta}{\cos\theta_o} \right]^6 \right]$$

UNKNOWNS:

$n_i(S)$ (i.e. DOUT)
Potential, P

INPUT:

N = total no. of ion and electron species

For each species
Z = charge no.
A = mass no.
T = $T_\perp$ = measured temperature
ANIS = $T_\perp/T_\parallel$ (guessed or varied)
DIN = n(S_o) = density at equator (or s/c), measured.

Geometry at equator (or s/c)
RSC = radial distance (in R_J)
CLSC = θ_C = centifugal latitude
GLSC = θ_G = Jovigraphic latitude (System III)
BLSC = θ_M = magnetic latitude

Geometry at S
R = radial distance (in R_J)
CL = centrifugal latitude
GL = Jovigraphic latitude
BL = magnetic latitude

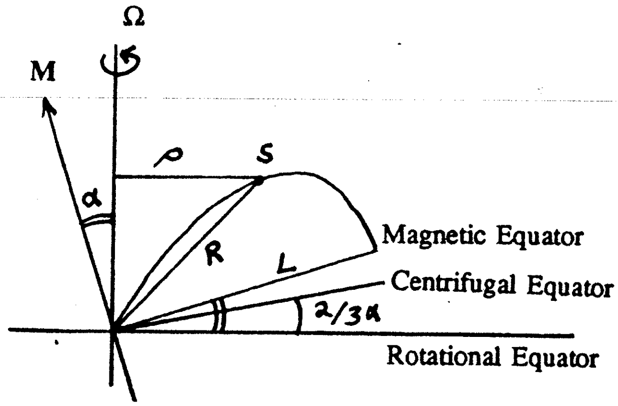
Also
IT = no. of iterations (10 usually o.k.)
FMIN = degree of charge neutrality required (say 1/1000 of the maximum density,i.e. ~ 1.0 for the hot torus).

The ambipolar potential is calculated numerically. The densities DOUT and charge neutrality F are calculated for an initial guess of P (i.e. P=0). The differential of F w.r.t. P, DF, is then calculated and P then changed. This procedure is iterated until F is less than FMIN.

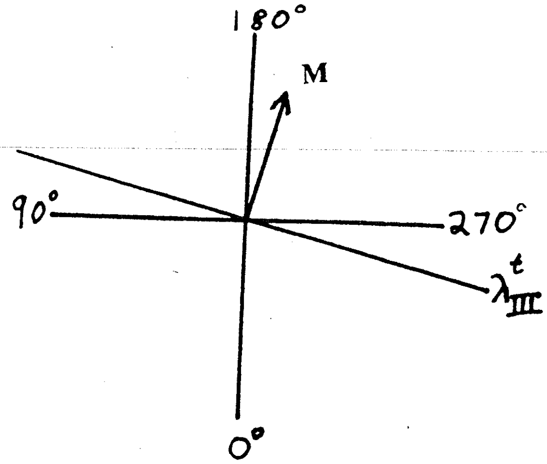
```
      SUBROUTINE SCAHGT(N,Z,A,T,ANIS,DIN,RSC,CLSC,
     *S3LON,GLSC,BLSC,R,CL,GL,BL,EPS,DOUT,IT,FMIN)
      REAL*8 E1,E2,E3,EE,F,DF,C1,C2,C
      DIMENSION Z(10),A(10),T(10),ANIS(10),DIN(10),DOUT(13)
      DATA DEGRAD/0.01745/,FACTOR/0.825/
c
c centrifugal force
c
      C2=R*R*COS(GL*DEGRAD)*COS(GL*DEGRAD)
      C1=RSC*RSC*COS(GLSC*DEGRAD)*COS(GLSC*DEGRAD)
      C=C1-C2
c
c magnetic mirror force
c
      F11=3.*SIN(BLSC*DEGRAD)*SIN(BLSC*DEGRAD)
      F21=3.*SIN(BL*DEGRAD)*SIN(BL*DEGRAD)
      F12=SQRT(1+F11)
      F22=SQRT(1+F21)
      F1=F12/(COS(BLSC*DEGRAD)**6)
      F2=F22/(COS(BL*DEGRAD)**6)
      F3=log(F2/F1)
c
c start iterations
c
      P=0.
      DO 10 K=1,IT
      F=0.
      DF=0.
c
c species
c
      DO 5 I=1,N
      DOUT(I)=0.
      E1=Z(I)*P/T(I)*ANIS(I)
      E2=FACTOR*A(I)*C/T(I)*ANIS(I)
      E3=(ANIS(I)-1)*F3
      EE=E1+E2-E3
      IF(DABS(EE).GT.75) GO TO 5
      DOUT(I)=DIN(I)*DEXP(EE)
c add up charge
      F=F+Z(I)*DOUT(I)
c calculate derivatives of F wrt P
      DF=DF-Z(I)*Z(I)*DOUT(I)/T(I)*ANIS(I)
  5   CONTINUE
      IF(DABS(F).LT.FMIN) GO TO 20
      IF(DF.EQ.0.) GO TO 20
c change potential P
      P=P-F/DF
 10   CONTINUE
 20   CONTINUE
      RETURN
      END
```

OFFSET TILTED DIPOLE AT JUPITER

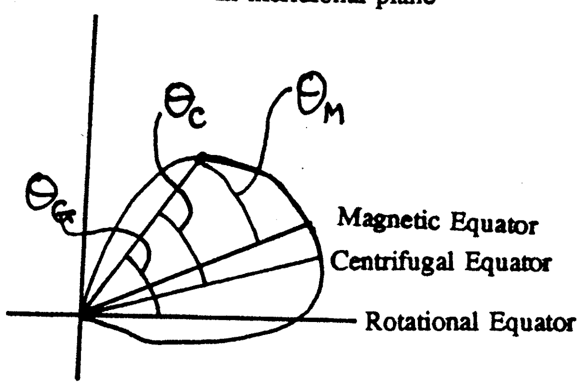
TILT



In meridional plane



In equatorial plane



$\sin \alpha \approx \sin \alpha_t \sin(\lambda_{III} - \lambda_{III}^{t})$

$\alpha_t = 9.6$ degrees

$\lambda_{III}^{t} = 292$ degrees

$\theta_M = \theta_G - \alpha$

$\theta_C = \theta_M + \alpha/3$

OFFSET

$L^O = L - \Delta \cos(\lambda_{III} - \lambda_{III}^{O})$

$L = R / \cos^2 \theta_M$

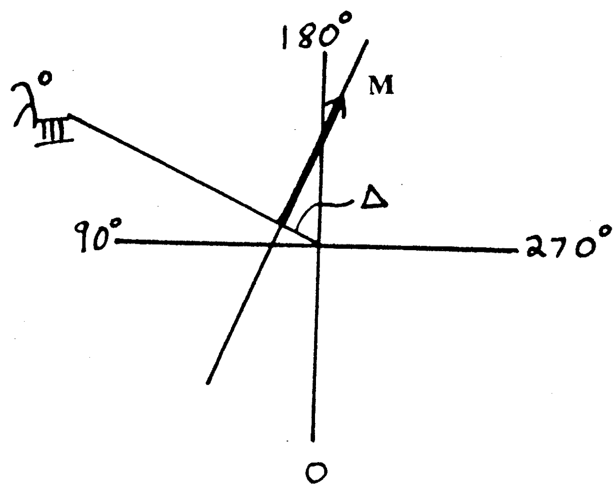
$\alpha = \sin^{-1} (\sin\alpha_t \sin(\lambda_{III} - \lambda_{III}^{t}))$

$\alpha_t = 9.6$ degrees

$\lambda_{III}^{t} = 292$ degrees

$\lambda_{III}^{O} = 148$ degress

$\Delta = 0.131\ R_J$



FLUXTUBE CONTENT FOR A DIPOLE L-SHELLS

Magnetic flux through L-shell of width ΔL is a constant along a given field line

$$\phi = B\,A\,\Delta L = \text{constant}$$
$$= \frac{B_o}{L^3}\,2\pi R_J^2\,L\Delta L \quad \text{at equator}$$

Using

$$B = \frac{B_o}{L^3}\frac{(1+3\sin^2\theta)^{1/2}}{\cos^6\theta}$$

and (see note (b) below)

$$ds = R_J\,L\cos\theta\,(1+3\sin^2\theta)^{1/2}d\theta$$

the total volume of a fluxtube width ΔL becomes

$$V = 2\int_{s=0}^{s_{max}} A\,\Delta L\,ds = 2\int_{s=0}^{s_{max}} \frac{\phi}{B}\,ds$$

$$= 2\int_{\theta=0}^{\theta_{max}} \frac{2\pi R_J^2\,L\Delta L\cos^6\theta}{(1+3\sin^2\theta)^{1/2}}\,R_J\,L\cos\theta\,(1+3\sin^2\theta)^{1/2}\,d\theta$$

$$= 4\pi R_J^3\,L^2\,\Delta L\int_{\theta=0}^{\theta_{max}}\cos^7\theta\,d\theta$$

Therefore the total number of ions per unit L-shell (ΔL = 1) is

$$N = 4\pi\,R_J^3\,L^2\int_{\theta=0}^{\theta_{max}} n\cos^7\theta\,d\theta$$

and

$$NL^2 = 4\pi\,R_J^3\,L^4\int_{\theta=0}^{\theta_{max}} n\cos^7\theta\,d\theta$$

In practical units this is numerically intergrated over ion species i and latitude steps j

$$NL^2 = \left[\underset{\uparrow\,m^3}{4\pi\,R_J^3}\,L^4\,10^6\frac{360}{2\pi}\right]\sum_{i,j}\underset{\uparrow\,cm^{-3}}{n_{ij}}\cos^7\theta_j\,\underset{\uparrow\,degrees}{\Delta\theta_j}$$

$$= 8 \times 10^{28}\,L^4 \sum_{i,j} n_{ij}\cos^7\theta_j\,\Delta\theta_j$$

If the plasma distribution is not symmetrically distributed about the magnetic equator then the north and south latitudes must be integrated separately and hence

$$NL^2 = 4x10^{28}\ L^4 \sum_{ij} n_{ij} \cos^7\theta_j\ \Delta\theta_j$$

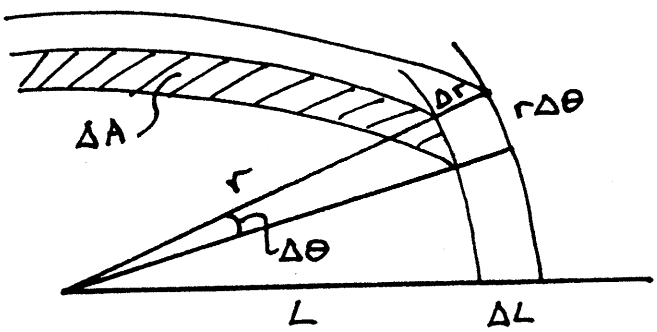
over $\pm\ \theta_{max}$ degrees of latitude.

Note:

(1) The approximation $\Delta V \approx \Delta A\ \Delta L \approx (2\pi r)\ (r\Delta\theta)\ \Delta r$ gives

$$V = 4\pi\ R_J^3\ L^2 \sum_j \cos^6\theta_j\ \Delta\theta_j$$

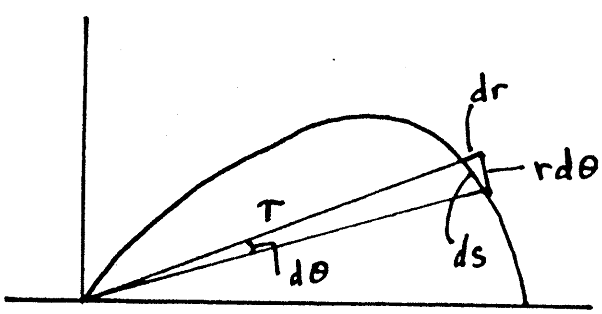
(because it ignores the curvature of the element of volume ΔV).



(2) Distance s along a dipole field line as a function of latitude θ

$$\begin{aligned} ds^2 &= dr^2 + r^2 d\theta^2 \\ &= (4L^2\sin^2\theta\cos^2\theta + L^2\cos^4\theta)\ d\theta^2 \\ &= (3\sin^2\theta + 1)\ L^2\cos^2\theta\ d\theta^2 \end{aligned}$$

hence $ds = L\cos\theta\ (1 + 3\sin^2\theta)\ d\theta$.



$r = L\cos^2\theta$

$dr = 2L\sin\theta\cos\theta\ d\theta$

Plasma Densities in the Io Plasma Torus

Given a point measurement in the Io plasma torus, the density of ion and electron species can be calculated everywhere along the given magnetic field line by solving a simple force balance equation.

$$(\nabla \cdot \mathbf{p})_{||} + (F_c)_{||} + n_i Z_i q E_{||} = 0 \tag{1}$$

where $\mathbf{p}$ is the gyrotropic pressure tensor, $(F_c)_{||}$ is the parallel component of the centrifugal force in the non-inertial rotating reference frame, and $E_{||}$ is the ambipolar electric field.

The first term is evaluated by rewriting the pressure tensor as

$$\mathbf{p} = \begin{pmatrix} p_\perp & 0 & 0 \\ 0 & p_\perp & 0 \\ 0 & 0 & p_\perp \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p_{||} - p_\perp \end{pmatrix} \tag{2}$$

So

$$\mathbf{p} = p_\perp \mathbf{I} + (p_{||} - p_\perp)\mathbf{bb} \tag{3}$$

where $\mathbf{I}$ is the unit tensor, is a dyadic formed from the unit vectors $\mathbf{b}$ along the magnetic field direction.

The divergence of the pressure tensor is

$$\begin{aligned} \nabla \cdot \mathbf{p} &= \nabla \cdot p_\perp \mathbf{I} + \nabla \cdot (p_{||} - p_\perp)\mathbf{bb} \\ &= \nabla p_\perp + (p_{||} - p_\perp)\nabla \cdot \mathbf{bb} + \mathbf{b}(\mathbf{b} \cdot \nabla)(p_{||} - p_\perp) \\ &= (\nabla_\perp + \nabla_{||})p_\perp + (p_{||} - p_\perp)[(\mathbf{b} \cdot \nabla)\mathbf{b} + \mathbf{b}(\nabla \cdot \mathbf{b})] + \nabla_{||}(p_{||} - p_\perp) \\ &= \nabla_\perp p_\perp + (p_{||} - p_\perp)[(\mathbf{b} \cdot \nabla)\mathbf{b} + \mathbf{b}(\nabla \cdot \mathbf{b})] + \nabla_{||} p_{||} \end{aligned} \tag{4}$$

The parallel and perpendicular components are

$$(\nabla \cdot \mathbf{p})_\perp = \nabla_\perp p_\perp + (p_{||} - p_\perp)(\mathbf{b} \cdot \nabla)\mathbf{b} \tag{5}$$

$$(\nabla \cdot \mathbf{p})_{||} = \nabla_{||} p_{||} + (p_{||} - p_\perp)\mathbf{b}(\nabla \cdot \mathbf{b}) \tag{6}$$

Notice that since $(\mathbf{b}\cdot\nabla)\mathbf{b} = (\nabla \times \mathbf{b}) \times \mathbf{b} + \nabla(b^2/2) = (\nabla \times \mathbf{b}) \times \mathbf{b}$, this term is always perpendicular to $\mathbf{b}$. In second term in the parallel component is evaluated as follows,

$$\begin{aligned} \mathbf{b}(\nabla \cdot \mathbf{b}) &= \mathbf{b}\left(\nabla \cdot \frac{\mathbf{B}}{B}\right) \\ &= \mathbf{b}\left[\frac{B(\nabla \cdot \mathbf{B}) - (\mathbf{B} \cdot \nabla)B}{B^2}\right] \\ &= -\frac{\mathbf{b}}{B^2}(\mathbf{B} \cdot \nabla)B \\ &= -\frac{\mathbf{b}}{B^2}B\frac{\partial B}{\partial s} \\ &= -\frac{\mathbf{b}}{B}\frac{\partial B}{\partial s} \end{aligned} \tag{7}$$

So

$$(\nabla \cdot \mathbf{p})_{||} = \nabla_{||}p_{||} - (p_{||} - p_{\perp})\frac{1}{B}\frac{\partial B}{\partial s} = \nabla_{||}p_{||} - (p_{||} - p_{\perp})\frac{\partial}{\partial s}\log B \tag{8}$$

The second term in the force balance equation is the centrifugal force directed along the magnetic field line. The centrifugal force is

$$F_c = \frac{n_i m_i v^2}{\rho} = \frac{n_i m_i \Omega^2 \rho^2}{\rho} = n_i m_i \Omega^2 \rho \tag{9}$$

where Ω is the angular velocity of the corotating plasma and ρ is the perpendicular distance to the spin axis. The centrifugal potential is

$$\phi_c = -\int F_c d\rho = -\int n_i m_i \Omega^2 \rho d\rho = -\frac{1}{2} n_i m_i \Omega^2 \rho^2 \tag{10}$$

By definition,

$$(F_c)_{||} = -\nabla_{||}\phi_c = -n_i m_i \frac{\partial}{\partial s}\left(\frac{1}{2}\Omega^2\rho^2\right) \tag{11}$$

The final term is the ambipolar electric field. This term is by definition

$$n_i Z_i q E_{||} = -n_i Z_i q \frac{\partial \Phi}{\partial s} \tag{12}$$

The final expression for the force balance equation is

$$\frac{\partial p_{||}}{\partial s} - (p_{||} - p_{\perp})\frac{\partial}{\partial s}\log B - n_i m_i \frac{\partial}{\partial s}\left(\frac{1}{2}\Omega^2\rho^2\right) - n_i Z_i q \frac{\partial \Phi}{\partial s} = 0 \tag{13}$$

Assume $p_{||} = n_i T_{||}$ and that $T_{||}$ is constant along the magnetic field line.

$$\frac{\partial n_i T_{||}}{\partial s} = n_i (T_{||} - T_{\perp})\frac{\partial}{\partial s}\log(B) + n_i m_i \frac{\partial}{\partial s}\left(\frac{1}{2}\Omega^2\rho^2\right) + n_i Z_i q \frac{\partial \Phi}{\partial s} \tag{14}$$

or

$$\frac{1}{n_i}\frac{\partial n_i}{\partial s} = \frac{(T_{||} - T_{\perp})}{T_{||}}\frac{\partial}{\partial s}\log(B) + \frac{m_i}{T_{||}}\frac{\partial}{\partial s}\left(\frac{1}{2}\Omega^2\rho^2\right) + \frac{Z_i q}{T_{||}}\frac{\partial \Phi}{\partial s} \tag{15}$$

or

$$\frac{\partial}{\partial s}\log n_i = \frac{\partial}{\partial s}\left[\left(1 - \frac{T_{\perp}}{T_{||}}\right)\log(B) + \frac{1}{2}\frac{m_i \Omega^2 \rho^2}{T_{||}} + \frac{Z_i q \Phi}{T_{||}}\right] \tag{16}$$

Integrating along the field line from a reference point S_o to S we have

$$\log\frac{n_i(S)}{n_i(S_o)} = \left[\left(1 - \frac{T_{\perp}}{T_{||}}\right)\log\frac{B(S)}{B(S_o)} + \frac{1}{2}\frac{m_i \Omega^2[\rho^2(S) - \rho^2(S_o)]}{T_{||}} + \frac{Z_i q[\Phi(S) - \Phi(S_o)]}{T_{||}}\right] \tag{17}$$

or

$$n_i(S) = n_i(S_o)\exp\left[\left(1 - \frac{T_{\perp}}{T_{||}}\right)\log\frac{B(S)}{B(S_o)} + \frac{1}{2}\frac{m_i \Omega^2[\rho^2(S) - \rho^2(S_o)]}{T_{||}} + \frac{Z_i q[\Phi(S) - \Phi(S_o)]}{T_{||}}\right] \tag{18}$$

Kappa Distribution

Now consider a kappa distribution rather than a Maxwell-Boltzmann distribution. For a kappa distribution [*Meyer-Vernet et al.*, 1995]

$$T_i \propto n_i^{\gamma-1} \tag{19}$$

where $\gamma = 1 - 1/(\kappa - 1/2)$ and κ generally lie in the range 2-6. The isotropic pressure is

$$p = \alpha_i n_i n_i^{\gamma-1} = \alpha_i n_i^{\gamma} \tag{20}$$

where α_i is the constant of proportionality (which can be determined from the initial measured temperature and density).

The force balance equation for an isotropic velocity distribution is

$$\frac{\partial \alpha_i n_i^{\gamma}}{\partial s} = n_i m_i \frac{\partial}{\partial s}\left(\frac{1}{2}\Omega^2 \rho^2\right) + n_i Z_i q \frac{\partial \Phi}{\partial s} \tag{21}$$

or

$$\alpha_i \gamma n_i^{\gamma-2} \frac{\partial n_i}{\partial s} = m_i \frac{\partial}{\partial s}\left(\frac{1}{2}\Omega^2 \rho^2\right) + Z_i q \frac{\partial \Phi}{\partial s} \tag{22}$$

Integrating along the magnetic field line from S to S_o

$$n_i(S) = \left\{ n_i^{\gamma-1}(S_o) + \left(\frac{\gamma-1}{2\alpha_i\gamma}\right) m_i \Omega^2 \left[\rho^2(S) - \rho^2(S_o)\right] + \left(\frac{\gamma-1}{\alpha_i\gamma}\right) Z_i q \left[\Phi(S) - \Phi(S_o)\right] \right\}^{\frac{1}{\gamma-1}} \tag{23}$$