

Voyager PLS Error Analysis

Errors in the Data Measurements

For all purposes in this Error Analysis memo, it will be assumed that currents are measured and displayed in femtoamps. This is the natural range for the measured currents in the Jovian system from the Voyager PLS instrument.

From an early voyager memo:

$$Error^2 = Error^2_{background} + Error^2_{Measurement}$$

$$Error^2_{background} = 54$$
$$Error^2_{Measurement} = 1.118 \times 10^{-4} I^2$$

In re-analysis of the code, this error seemed extremely small and did not even appear visible, as is obvious in Figure 1. All figures are included in Appendix A. Because of the range of these currents, a natural logarithmic scale is used. At some point in this scale, there should be a visual error, even if not at all points. So obviously, the ~7 femtoamp background noise (square root of 54) does not do an adequate job of accounting for noise in the data.

In order to solve this problem, the residuals from a fit IV curve were used, as shown in Figure 2. If the fit is a good fit, then the residuals should help to determine the true background noise. Similarly we determined a rough estimate for the background noise by eye. We made 2 attempted estimates of the error. The larger estimate took an estimated background noise of 1,000 femtoamps and a 1% error in the measurements. The smaller estimate assumed a background noise of 300 femtoamps and a 0.5% error in the measurements. From these, the two errors below were derived:

$$A) Error^2 = \sqrt{10^6 + 1.0 \times 10^{-4} I^2}$$

$$B) Error^2 = \sqrt{10^5 + 2.5 \times 10^{-5} I^2}$$

Examples for the same two spectra were included in Figure 3 for both errors and A and B. From the given spectra it would appear that error method B is a more accurate method for determining the errors in the measured data from the PLS instrument.

Poisson counting statistics can also be taken into account. Assuming a worse case scenario where all of the ions are doubly charged then it is possible to calculate the

number of ion impacts on a plate and the Poisson measurement error in the process, see Rob Wilson's calculation in Appendix A. This generates a little bit more error giving the following. Here, t is the accumulation period (in seconds) per channel of the PLS instrument. The accumulation time is typically 0.24 seconds for the Voyager PLS Faraday cup:

$$C) \text{ Error}^2 = \sqrt{10^5 + 2.5 \times 10^{-5} I^2 + \frac{0.0179}{t} I}$$

Figure 3 also contains the C error to show that there is no visual difference between error B and C due to the addition of a Poisson-like term. Considering the PLS is a Faraday cup and measured current and did not measure Poisson error, and that it did not even make a visual, and hardly a numerical, difference, this error was not used for the calculation and measurement error B was the error used for analysis at Jupiter. Since it accounts for the background noise, this error may change for analysis at other planets or in the solar wind.

Uncertainties in the Parameters

In order to fit the model to the data, Alan Barnett's response function was used. This model is fit to the data with the assistance of a function in IDL called MPFIT, written by Craig Markwardt. Using the above error of B in MPFIT allows for the program to accurately minimize the chi-square value of the best fit parameters for any number of parameters. Usually, the fitting routine fits 3 directional flows of the velocity, the temperature of the ion species, and the densities of 3 or 4 species for our model.

In our experience, MPFIT does not converge to the true minimum, but gets extremely close. Therefore, after leaving the fitting routine, a check is made that the fit is at the true best fit location, making slight final adjustments to the parameters and getting them into the true best fit location as defined by minimizing the Chi-Squared value.

$$\chi^2 = \sum_{i=1}^N \frac{(Data_i - Model_i)^2}{Error_i^2}$$

Once the true minimum is found then the uncertainties in the parameters can be derived. First, the Hessian matrix is calculated which describes the local curvature of a function of many variables. The size of the matrix is v by v , where v is the number of free parameters. Each element of the Hessian matrix can be described below:

$$H_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

From this, it is possible to construct the entire Hessian matrix. To calculate the one used in this analysis, the ∂x steps were relative step sizes of 1% of the best-fit parameters. Therefore the wide range of magnitudes of parameters is accounted for. If an absolute step size were used then a moderate change in the temperature in the cold Torus, about 0.02 of 2 eV, would be a very small change in the density of S^+ , 0.02 of 1,200 n/cc. Therefore this percentage was used to reflect an accurate curvature for all of the elements. It was also evident that the Chi-Squared space was noisy at small absolute step levels and the curvature matrix at that point would not reflect the overall curvature of the function itself; this is due to noise in the data and the co-dependence of the parameters in the function.

From the Hessian matrix, the curvature matrix is created, which is the same as dividing the Hessian matrix by a factor of 2. To get the covariance matrix, it just requires the inverse of the curvature matrix. The square roots of the diagonals of this matrix are the formal 1σ uncertainties in the best-fit parameters.

Because this is in Chi-Squared space and not reduced Chi-Squared space, the curvature matrix does not represent the true curvature of the function, but rather a function scaled by the degrees of freedom (DOF). Therefore, from the channels that are fit in the fitting routine we can define the DOF, where N is the number of channels being fit and ν is the number of free parameters.

$$DOF = N - \nu$$

Since the code returns the un-scaled 1σ uncertainties from the covariance matrix, it just requires a scaling factor of the square root of the DOF to account for it. The code for calculating the 1σ uncertainties comes from Wilson's paper "Error Analysis for numerical estimates of space plasma parameters". The only modification to the code is that the Δ in calculating the curvature matrix is not an absolute value but rather a relative value that depends on value of the best-fit parameter.

$$\sigma = 1\sigma_{un-scaled} * \sqrt{DOF}$$

$$parameters = parameter_{best} \pm \sigma$$

The best fit parameters in the above equation are returned directly from the fitting routine.

Reduced Chi-Squared Space

In order to test whether or not the fit parameters are the best, and to tell how they are dependent on each other, it is important to look at the reduced chi-squared space. This was done by taking the best-fit parameters and using the formal 1σ errors calculated above to create an array of reduced chi-square values. For this array, values of the reduced chi-squared are calculated within $\pm 3\sigma$ of the best-fit parameter.

$$\chi_R^2 = \frac{1}{N - \nu} \sum_{i=1}^N \frac{(Data_i - Model_i)^2}{Error_i^2}$$

N is the number of data points that the reduced chi-squared is being calculated for. For the formal analysis of these error points, data was selected carefully where the model matched the data. At points where the model goes to zero and there is just noise in the instrument, the chi-squared value was not calculated. The error value in the measurements that was used was the B measurement error from above.

These values are calculated for variations in the parameter space from the parameters that are fit only, in a high resolution between $\pm 3\sigma$.

Note: The IDL code returns a CSV with the best-fit parameters. Make sure to use this unedited version of the CSV file to generate the Chi contour plots. If edits are made to this CSV file it appears that IDL encounters rounding errors and will displace the true minimum of the fit. Therefore if no changes are made to the actual CSV file for the parameters of interest, the chi-contours should produce the correct plots.

To illustrate, here is the following case with 2 best fit parameters. If the fit were to return the following values for T_i and V_φ :

$$\begin{aligned} T_i &= 4.3 \pm 0.3 \text{ eV} \\ V_\varphi &= 60 \pm 1 \text{ km/s} \end{aligned}$$

The following grid represents the values at the corners of the plot:

$T_i = 3.4, V_\varphi = 57$	$T_i = 4.3, V_\varphi = 57$	$T_i = 5.2, V_\varphi = 57$
$T_i = 3.4, V_\varphi = 60$	$T_i = 4.3, V_\varphi = 60$	$T_i = 5.2, V_\varphi = 60$
$T_i = 3.4, V_\varphi = 63$	$T_i = 4.3, V_\varphi = 63$	$T_i = 5.2, V_\varphi = 63$

In between these values, many additional values are calculated to create a smooth contour plot of the reduced-chi squared values. For the plot, to put all of the parameters around the same value, we plot the delta of the reduced-chi squared value. Therefore, the actual value plotted is the reduced chi-squared value minus the minimum value of the reduced chi-squared value. Therefore, the minimum of every plot is by definition 0.

In addition, on the plots the 1σ uncertainties in the parameters are plotted in white error bars. Accounting for a not perfectly quadratic reduced chi-squared minimum, the values of 1 , 2 , and $3\sigma_{\Delta}$ are plotted for the delta values. Assuming a model with one independent parameter, these values would respectively be 1 , 4 , and 9 . For multiple parameters, a table is attached at the end to show the appropriate values to use. This does not assume that the chi-squared values near the minimum form a perfect parabolic shape. On all of the contour plots, these values are plotted in red dashed lines. The equation used to solve for these contour levels of delta is below.

$$\Delta = 2 * \text{Inverse of incomplete gamma}(p, \frac{\nu}{2})$$

Here, p is the confidence level, 0.6827 for $1\sigma_{\Delta}$, 0.954 for $2\sigma_{\Delta}$, and 0.997 for $3\sigma_{\Delta}$. These represent the 68% , 95% , and 99.7% confidence levels that the true value is within that value of σ_{Δ} . The number of parameters is denoted by the Greek letter ν .

Example contour plots for the 2 spectra are included in Figure 4. All of these plots were created in the Matlab coding language, as that language has better built in functionality for plotting contours. All data was created in the IDL language.

Appendix A

Figure 1

Two different plots with errors from the original Voyager Memo

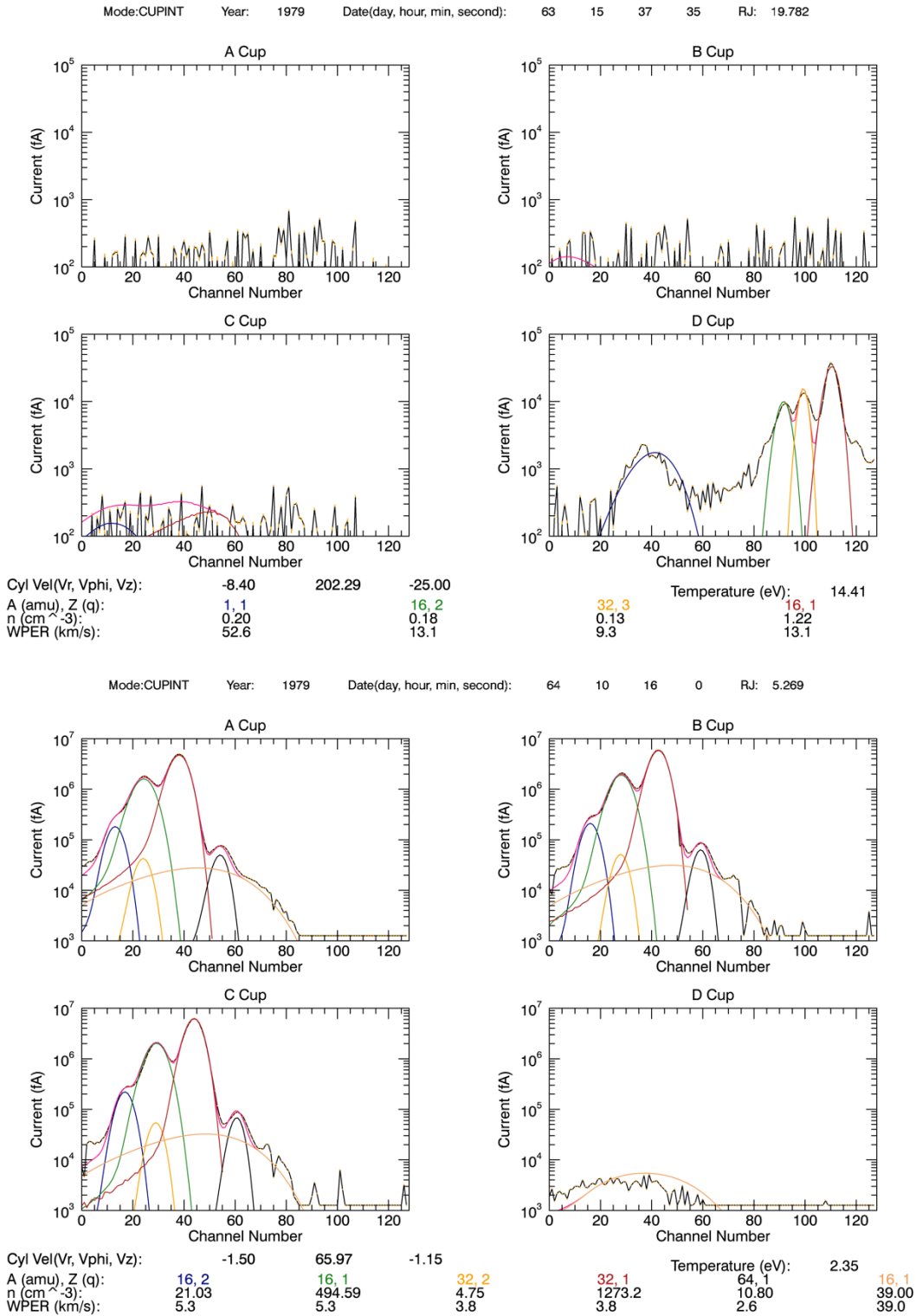


Figure 2

Residuals for the two fit spectra from Figure 1, in the same order

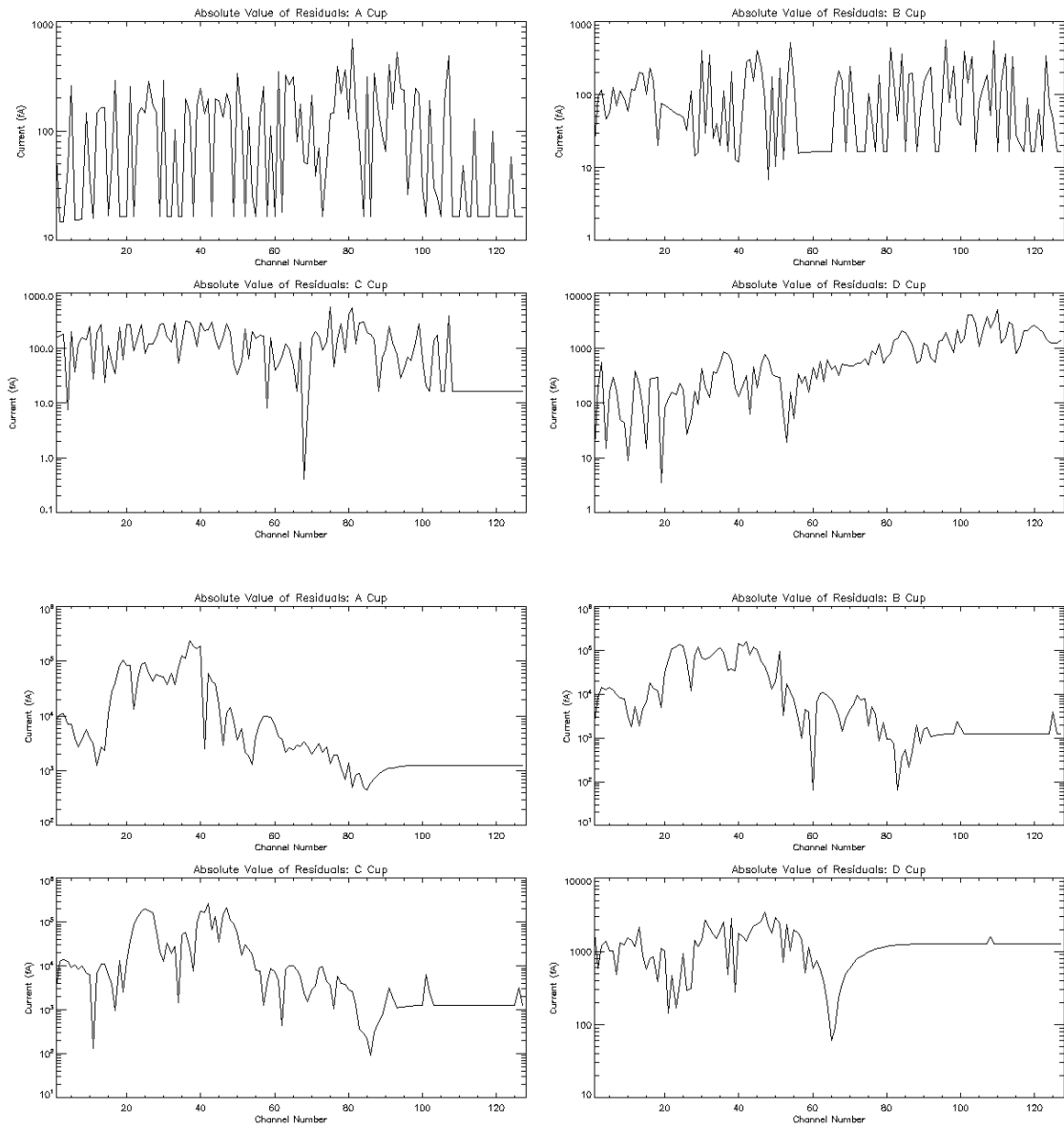
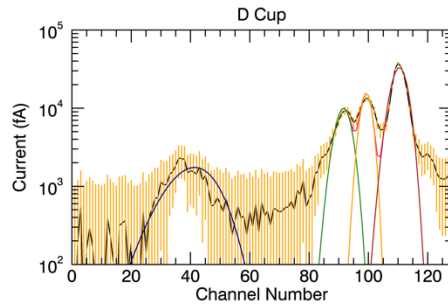
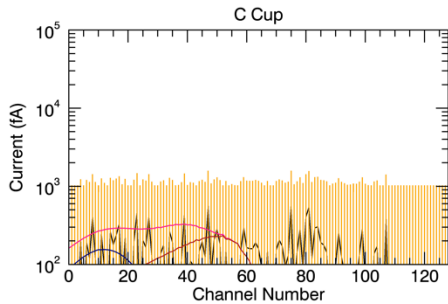
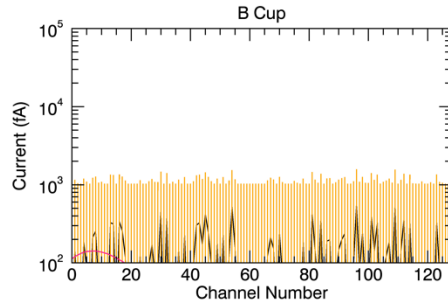
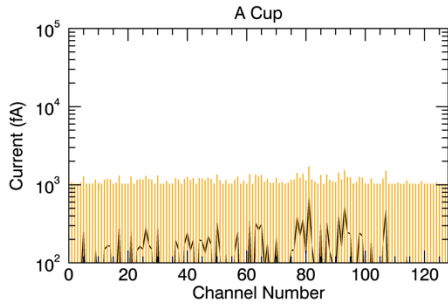


Figure 3

Comparison between errors A and B for the same two spectra

MEASUREMENT ERROR A

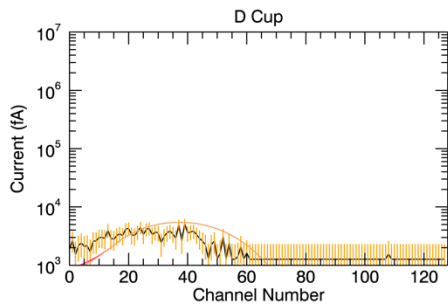
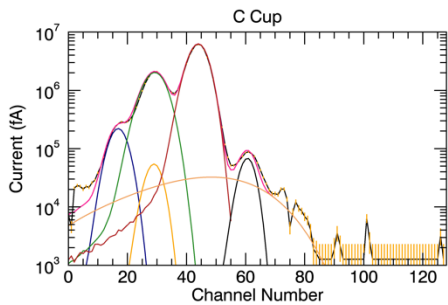
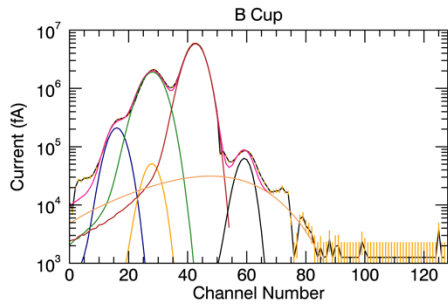
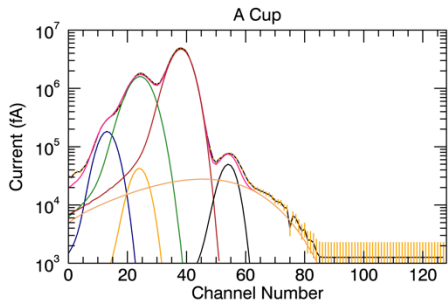
Mode:CUPINT Year: 1979 Date(day, hour, min, second): 63 15 37 35 RJ: 19.782



Cyl Vel(Vr, Vphi, Vz): -8.40 202.29 -25.00
 A (amu), Z (q): 1, 1 16, 2
 n (cm⁻³): 0.20 0.18
 WPER (km/s): 52.6 13.1

Temperature (eV): 32.3 16.1 14.41
 0.13 1.22
 9.3 13.1

Mode:CUPINT Year: 1979 Date(day, hour, min, second): 64 10 16 0 RJ: 5.269

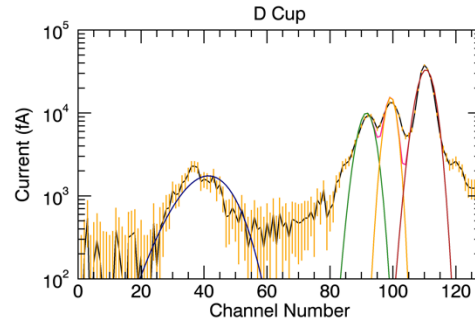
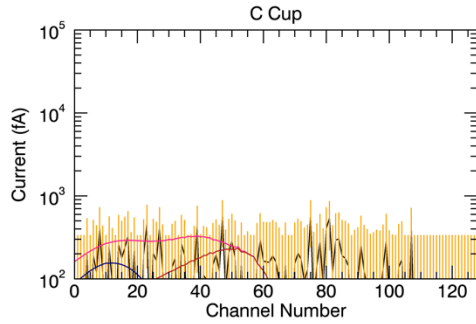
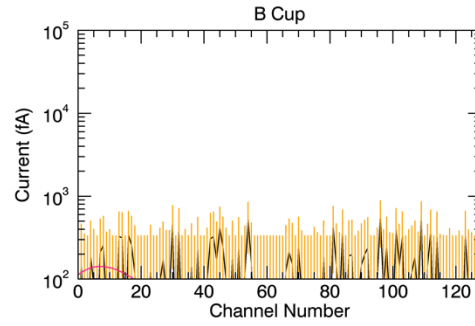
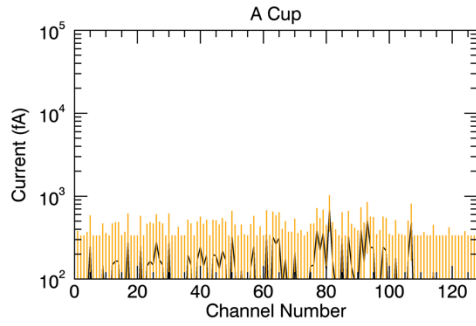


Cyl Vel(Vr, Vphi, Vz): -1.50 65.97 -1.15
 A (amu), Z (q): 16, 2 16, 1 32, 2
 n (cm⁻³): 21.03 494.59 4.75
 WPER (km/s): 5.3 5.3 3.8

Temperature (eV): 32.1 64.1 2.35 16.1 39.00
 1273.2 10.80
 3.8 2.6 39.0

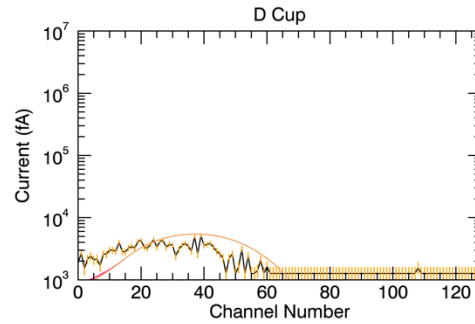
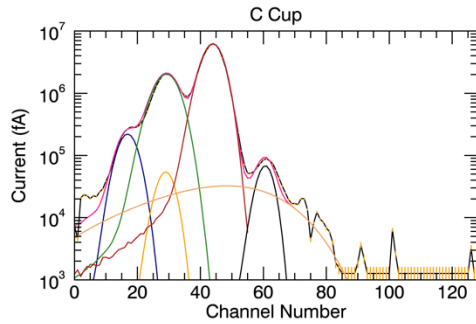
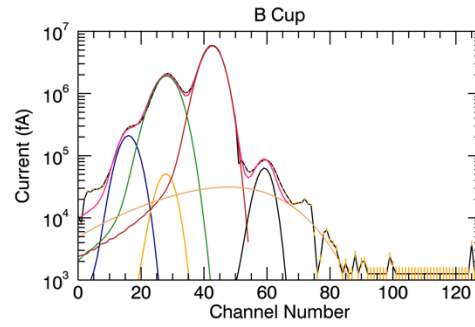
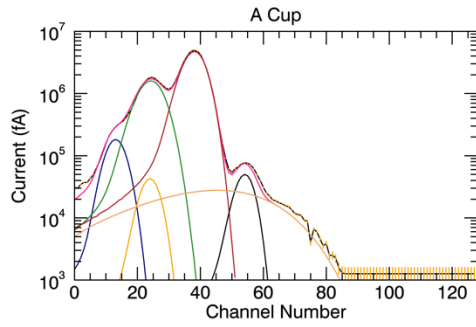
MEASUREMENT ERROR B

Mode:CUPINT Year: 1979 Date(day, hour, min, second): 63 15 37 35 RJ: 19.782



Cyl Vel(Vr, Vphi, Vz):	-8.40	202.29	-25.00		
A (amu), Z (q):	1, 1		16, 2		
n (cm ⁻³):	0.20		0.18	32.3	Temperature (eV): 16.1
WPER (km/s):	52.6		13.1	0.13	14.41
				9.3	1.22
					13.1

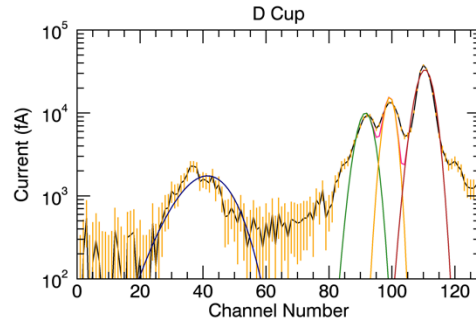
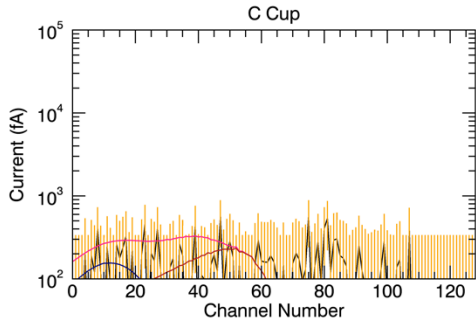
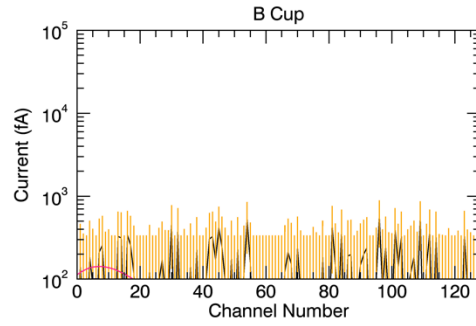
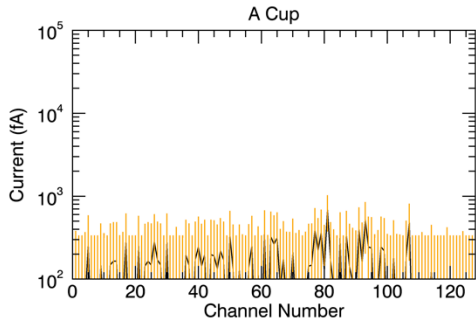
Mode:CUPINT Year: 1979 Date(day, hour, min, second): 64 10 16 0 RJ: 5.269



Cyl Vel(Vr, Vphi, Vz):		-1.50	65.97	-1.15	
A (amu), Z (q):	16, 2		16, 1	32, 2	
n (cm ⁻³):	21.03		494.59	4.75	Temperature (eV): 2.35
WPER (km/s):	5.3		5.3	3.8	64.1
					10.80
					2.6
					16.1
					39.0
					39.0

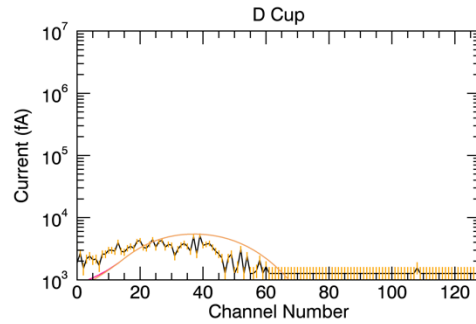
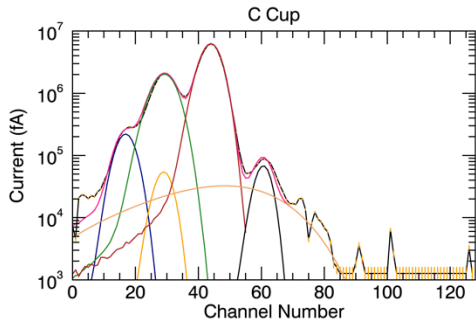
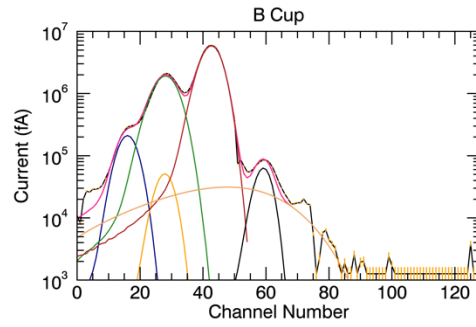
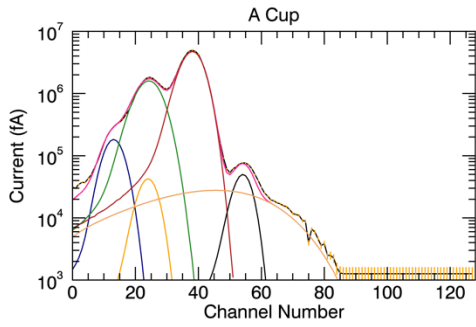
MEASUREMENT ERROR C

Mode:CUPINT Year: 1979 Date(day, hour, min, second): 63 15 37 35 RJ: 19.782



Cyl Vel(Vr, Vphi, Vz):	-8.40	202.29	-25.00		
A (amu), Z (q):	1, 1		16, 2		
n (cm ⁻³):	0.20		0.18	32.3	Temperature (eV): 16.1
WPER (km/s):	52.6		13.1	0.13	1.22
				9.3	13.1

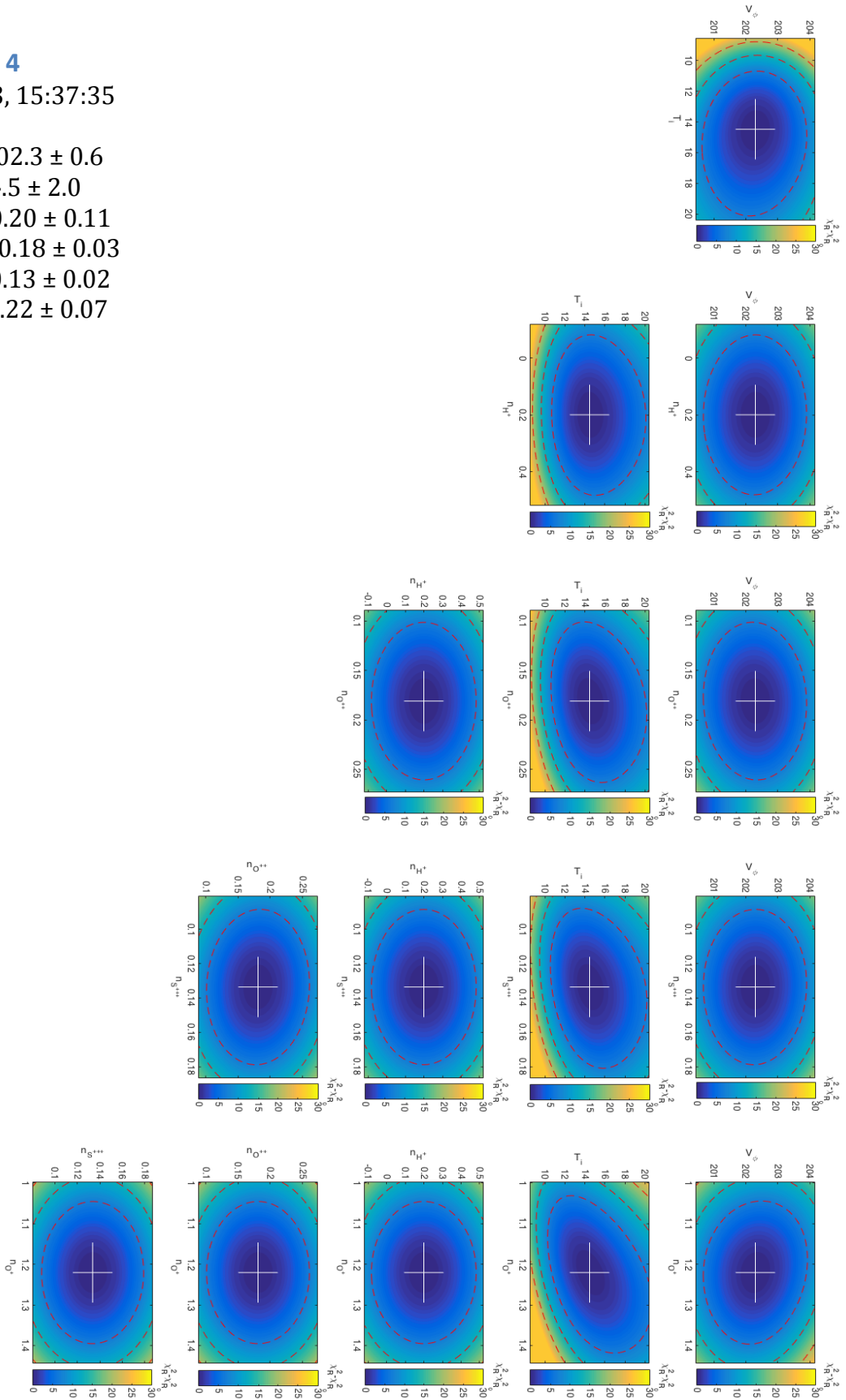
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Cyl Vel(Vr, Vphi, Vz):		-1.50	65.97	-1.15	
A (amu), Z (q):	16, 2		16, 1	32, 2	
n (cm ⁻³):	21.03		494.59	4.75	Temperature (eV): 64.1
WPER (km/s):	5.3		5.3	3.8	10.80
					2.6
					2.35
					16.1
					39.0
					39.0

Figure 4
Day 63, 15:37:35

$V_\phi = 202.3 \pm 0.6$
 $T_i = 14.5 \pm 2.0$
 $n_{H^+} = 0.20 \pm 0.11$
 $n_{O^{++}} = 0.18 \pm 0.03$
 $n_{O^+} = 0.13 \pm 0.02$
 $n_{S^+} = 1.22 \pm 0.07$



Day 64, 10:16:00

$$V_r = -1.45 \pm 0.06$$

$$V_\phi = 65.98 \pm 0.02$$

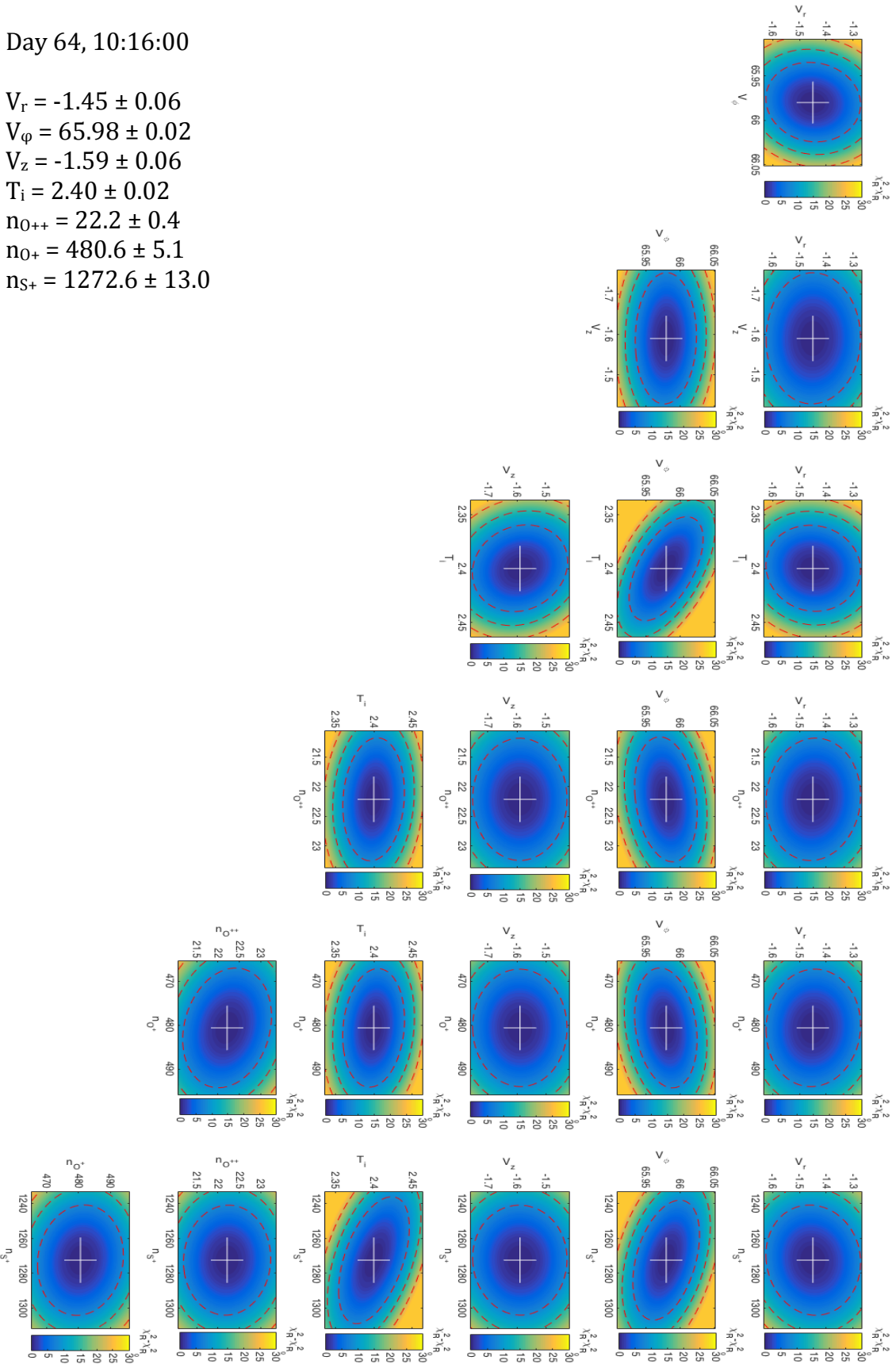
$$V_z = -1.59 \pm 0.06$$

$$T_i = 2.40 \pm 0.02$$

$$n_{0++} = 22.2 \pm 0.4$$

$$n_{0+} = 480.6 \pm 5.1$$

$$n_{S+} = 1272.6 \pm 13.0$$



Day 64, 10:16:00 – Fit with a reduced parameter space

$$V_\phi = 65.98 \pm 0.02$$

$$T_i = 2.40 \pm 0.02$$

$$n_{0++} = 22.2 \pm 0.4$$

$$n_{0+} = 480.5 \pm 5.1$$

$$n_{S+} = 1272.8 \pm 13.0$$

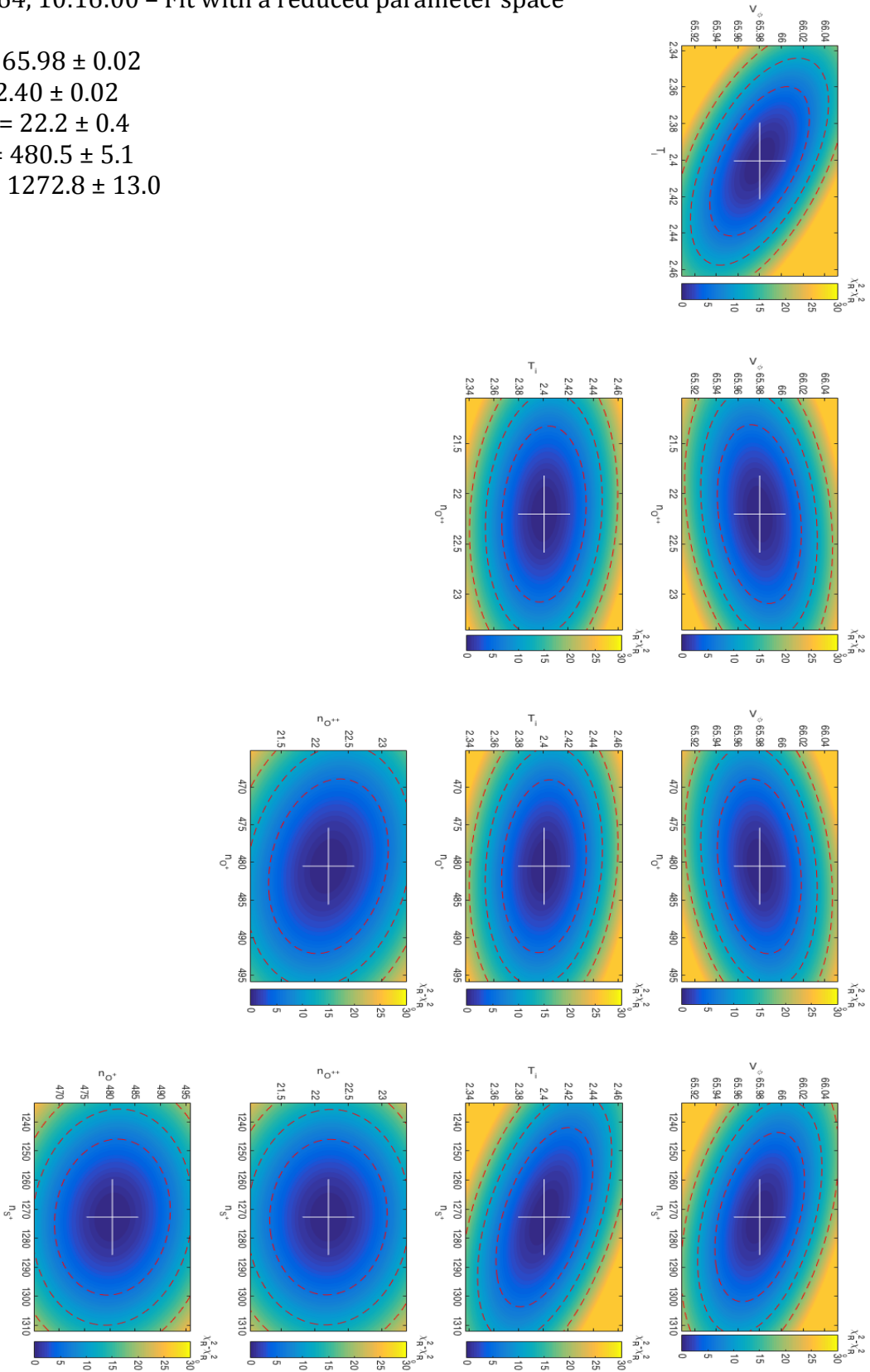


Table of Delta values for different number of free parameters. Up to 10 free parameters included in the table.

No. parameters	$1\sigma_{\Delta}$	$2\sigma_{\Delta}$	$3\sigma_{\Delta}$
1	1	4	9
2	2.29575	6.18007	11.8292
3	3.52674	8.02488	14.1564
4	4.71947	9.71563	16.2513
5	5.8876	11.3139	18.2053
6	7.0384	12.8488	20.0621
7	8.17624	14.3371	21.8466
8	9.30391	15.7891	23.5746
9	10.4234	17.2118	25.2569
10	11.536	18.6103	26.9011

Calculation of Poisson errors to follow on the next page